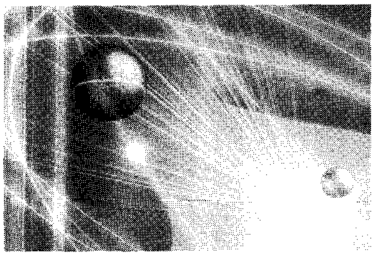


CHAPTER 10

Vectors and Coordinate Geometry in 3-Space

Introduction A complete real-variable calculus program involves the study of

(i) real-valued functions of a single real variable,

(ii) vector-valued functions of a single real variable,

(iii) real-valued functions of a real vector variable,

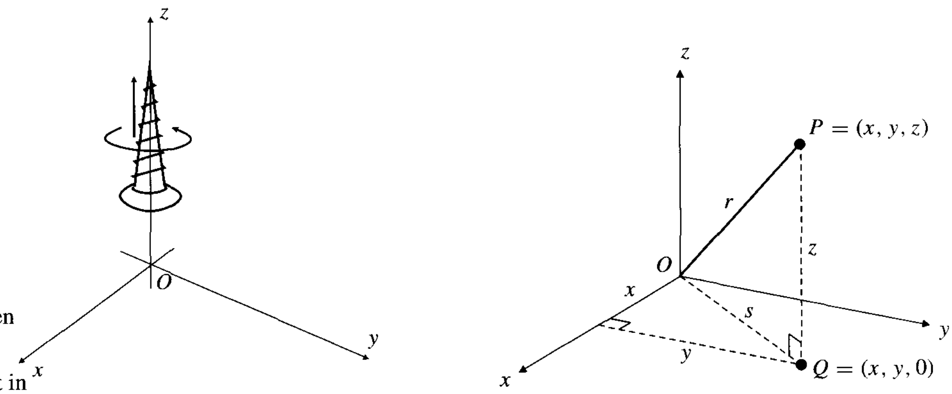
(iv) vector-valued functions of a real vector variable.

Chapters 1–9 are concerned with item (i). The remaining chapters deal with items (ii), (iii), and (iv). Specifically, Chapter 11 deals with vector-valued functions of a single real variable. Chapters 12–14 are concerned with the differentiation and integration of real-valued functions of several real variables, that is, of a real vector variable. Chapters 15 and 16 present aspects of the calculus of functions whose domains and ranges both have dimension greater than one, that is, vector-valued functions of a vector variable. Most of the time we will limit our attention to vector functions with domains and ranges in the plane, or in 3-dimensional space.

In this chapter we will lay the foundation for multivariable and vector calculus by extending the concepts of analytic geometry to three or more dimensions and by introducing vectors as a convenient way of dealing with several variables as a single entity. We also introduce matrices, because these will prove useful for formulating some of the concepts of calculus. This chapter is not intended to be a course in linear algebra. We develop only those aspects that we will use in later chapters and omit most proofs.

10.1 Analytic Geometry in 3 Dimensions

We say that the physical world in which we live is three-dimensional because through any point there can pass three, and no more, straight lines that are **mutually perpendicular**, that is, each of them is perpendicular to the other two. This is equivalent to the fact that we require three numbers to locate a point in space with respect to some reference point (the **origin**). One way to use three numbers to locate a point is by having them represent (signed) distances from the origin, measured in the directions of three mutually perpendicular lines passing through the origin. We call such a set of lines a Cartesian coordinate system, and each of the lines is called a coordinate axis. We usually call these axes the x-axis, the y-axis, and the z-axis, regarding the x- and y-axes as lying in a horizontal plane and the z-axis as vertical. Moreover, the coordinate system should have a **right-handed orientation**. This means that the thumb, forefinger, and middle finger of the right hand can be extended so as to point, respectively, in the directions of the positive x-axis, the positive y-axis, and the positive z-axis. For the more mechanically minded, a right-handed screw will advance in the positive z direction if twisted in the direction of rotation from the positive x-axis toward the positive y-axis. (See Figure 10.1(a).)



(a) (b)

Figure 10.1

(a) The screw moves upward when twisted counterclockwise as seen from above

(b) The three coordinates of a point in 3-space

With respect to such a Cartesian coordinate system, the **coordinates** of a point P in 3-space constitute an ordered triple of real numbers, (x, y, z). The numbers x, y, and z are, respectively, the signed distances of P from the origin, measured in the directions of the x-axis, the y-axis, and the z-axis. (See Figure 10.1(b).)

Let Q be the point with coordinates $(x, y, 0)$. Then Q lies in the xy-plane (the plane containing the x- and y-axes) directly under (or over) P. We say that Q is the vertical projection of P onto the xy-plane. If r is the distance from the origin O to P and s is the distance from O to Q, then, using two right-angled triangles, we have

$$s^2 = x^2 + y^2 \qquad \text{and} \qquad r^2 = s^2 + z^2 = x^2 + y^2 + z^2.$$

Thus, the distance from P to the origin is given by

$$r = \sqrt{x^2 + y^2 + z^2}.$$

Similarly, the distance r between points $P_1 = (x_1, y_1, z_1)$ and $P_2 = (x_2, y_2, z_2)$ (see Figure 10.2) is

$$r = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

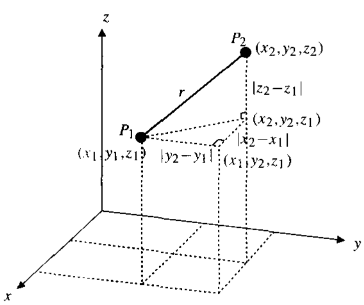


Figure 10.2

Example 1 Show that the triangle with vertices $A = (1, -1, 2)$, $B = (3, 3, 8)$, and $C = (2, 0, 1)$ has a right angle.

Solution We calculate the lengths of the three sides of the triangle:

$$a = |BC| = \sqrt{(2-3)^2 + (0-3)^2 + (1-8)^2} = \sqrt{59}$$
$$b = |AC| = \sqrt{(2-1)^2 + (0+1)^2 + (1-2)^2} = \sqrt{3}$$
$$c = |AB| = \sqrt{(3-1)^2 + (3+1)^2 + (8-2)^2} = \sqrt{56}$$

By the cosine law, $a^2 = b^2 + c^2 - 2bc\cos A$. In this case $a^2 = 59 = 3 + 56 = b^2 + c^2$, so that $2bc\cos A$ must be 0. Therefore $\cos A = 0$ and $A = 90°$.

■

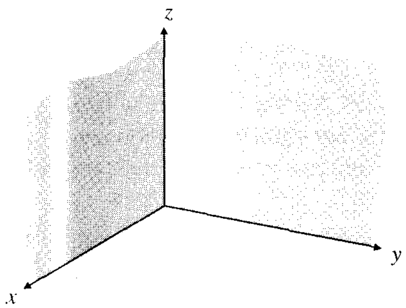


Figure 10.3 The first octant

Just as the x- and y-axes divide the xy-plane into four quadrants, so also the three **coordinate planes** in 3-space (the xy-plane, the xz-plane, and the yz-plane) divide 3-space into eight **octants**. We call the octant in which $x \geq 0$, $y \geq 0$, and $z \geq 0$ the **first octant**. When drawing graphs in 3-space it is sometimes easier to draw only the part lying in the first octant (Figure 10.3).

An equation or inequality involving the three variables x, y, and z defines a subset of points in 3-space whose coordinates satisfy the equation or inequality. A single equation usually represents a surface (a two-dimensional object) in 3-space.

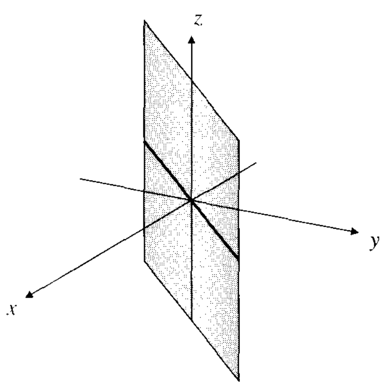


Figure 10.4 Equation $x = y$ defines a vertical plane

Example 2 **(Some equations and the surfaces they represent)**

(a) The equation $z = 0$ represents all points with coordinates $(x, y, 0)$, that is, the xy-plane. The equation $z = -2$ represents all points with coordinates $(x, y, -2)$, that is, the horizontal plane passing through the point $(0, 0, -2)$ on the z-axis.

(b) The equation $x = y$ represents all points with coordinates (x, x, z). This is a vertical plane containing the straight line with equation $x = y$ in the xy-plane. The plane also contains the z-axis. (See Figure 10.4.)

(c) The equation $x + y + z = 1$ represents all points the sum of whose coordinates is 1. This set is a plane that passes through the three points $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$. These points are not collinear (they do not lie on a straight line), so there is only one plane passing through all three. (See Figure 10.5.) The equation $x + y + z = 0$ represents a plane parallel to the one with equation $x + y + z = 1$ but passing through the origin.

(d) The equation $x^2 + y^2 = 4$ represents all points on the vertical circular cylinder containing the circle with equation $x^2 + y^2 = 4$ in the xy-plane. This cylinder has radius 2 and axis along the z-axis. (See Figure 10.6.)

(e) The equation $z = x^2$ represents all points with coordinates (x, y, x^2). This surface is a parabolic cylinder tangent to the xy-plane along the y-axis. (See Figure 10.7.)

(f) The equation $x^2 + y^2 + z^2 = 25$ represents all points (x, y, z) at distance 5 from the origin. This set of points is a *sphere* of radius 5 centred at the origin. ■

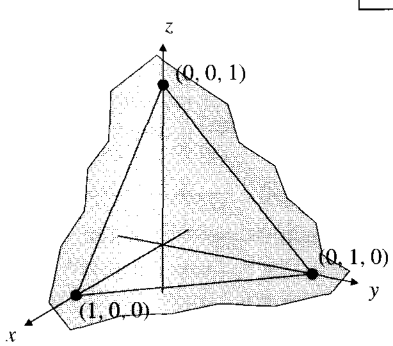


Figure 10.5 The plane with equation $x + y + z = 1$

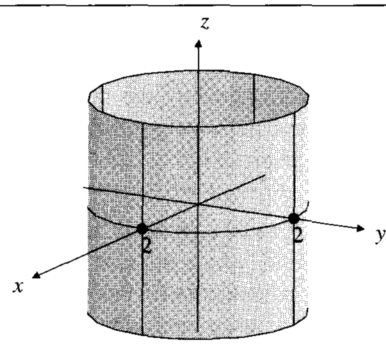


Figure 10.6 The circular cylinder with equation $x^2 + y^2 = 4$

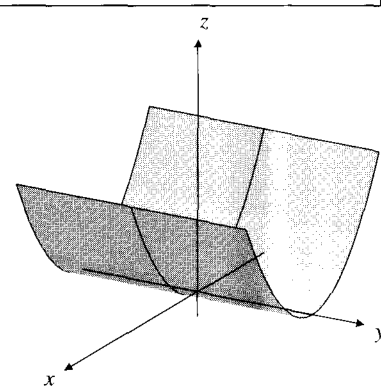


Figure 10.7 The parabolic cylinder with equation $z = x^2$

Observe that equations in x, y, and z need not involve each variable explicitly. When one of the variables is missing from the equation, the equation represents a surface *parallel to* the axis of the missing variable. Such a surface may be a plane or a cylinder. For example, if z is absent from the equation, the equation represents in 3-space a vertical (i.e., parallel to the z-axis) surface containing the curve with the same equation in the xy-plane.

Occasionally a single equation may not represent a two-dimensional object (a surface). It can represent a one-dimensional object (a line or curve), a zero-dimensional object (one or more points), or even nothing at all.

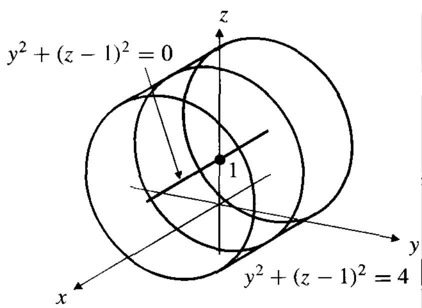


Figure 10.8 The cylinder $y^2 + (z-1)^2 = 4$ and its axial line $y^2 + (z-1)^2 = 0$

Example 3 Identify the graphs of the equations: (a) $y^2 + (z-1)^2 = 4$, (b) $y^2 + (z-1)^2 = 0$, (c) $x^2 + y^2 + z^2 = 0$, and (d) $x^2 + y^2 + z^2 = -1$.

Solution

(a) Since x is absent, the equation $y^2 + (z-1)^2 = 4$ represents an object parallel to the x-axis. In the yz-plane the equation represents a circle of radius 2 centred at $(y, z) = (0, 1)$. In 3-space it represents a horizontal circular cylinder, parallel to the x-axis, with axis one unit above the x-axis. (See Figure 10.8.)

(b) Since squares cannot be negative, the equation $y^2 + (z-1)^2 = 0$ implies that $y = 0$ and $z = 1$, so it represents points $(x, 0, 1)$. All these points lie on the line parallel to the x-axis and one unit above it. (See Figure 10.8.)

(c) As in part (b), $x^2 + y^2 + z^2 = 0$ implies that $x = 0$, $y = 0$, and $z = 0$. The equation represents only one point, the origin.

(d) The equation $x^2 + y^2 + z^2 = -1$ is not satisfied by any real numbers x, y, and z, so it represents no points at all. ■

A single inequality in x, y, and z typically represents points lying on one side of the surface represented by the corresponding equation (together with points on the surface if the inequality is not strict).

Example 4

(a) The inequality $z > 0$ represents all points above the xy-plane.

(b) The inequality $x^2 + y^2 \ge 4$ says that the square of the distance from (x, y, z) to the nearest point $(0, 0, z)$ on the z-axis is at least 4. This inequality represents all points lying on or outside the cylinder of Example 2(d).

(c) The inequality $x^2 + y^2 + z^2 \le 25$ says that the square of the distance from (x, y, z) to the origin is no greater than 25. It represents the solid ball of radius 5 centred at the origin, which consists of all points lying inside or on the sphere of Example 2(f). ■

Two equations in x, y, and z normally represent a one-dimensional object, the line or curve along which the two surfaces represented by the two equations intersect. Any point whose coordinates satisfy both equations must lie on both the surfaces, so must lie on their intersection.

Example 5 What sets of points in 3-space are represented by the pairs of equations?

(a) $\begin{cases} x + y + z = 1 \\ y - 2x = 0 \end{cases}$ (b) $\begin{cases} x^2 + y^2 + z^2 = 1 \\ x + y = 1 \end{cases}$

Solution

(a) The equation $x + y + z = 1$ represents the oblique plane of Example 2(c), and the equation $y - 2x = 0$ represents a vertical plane through the origin and the point $(1, 2, 0)$. Together these two equations represent the line of intersection of the two planes. This line passes through, for example, the points $(0, 0, 1)$ and $(\frac{1}{3}, \frac{2}{3}, 0)$. (See Figure 10.9(a).)

(b) The equation $x^2 + y^2 + z^2 = 1$ represents a sphere of radius 1 with centre at the origin, and $x + y = 1$ represents a vertical plane through the points $(1, 0, 0)$ and $(0, 1, 0)$. The two surfaces intersect in a circle, as shown in Figure 10.9(b). The line from $(1, 0, 0)$ to $(0, 1, 0)$ is a diameter of the circle, so the centre of the circle is $(\frac{1}{2}, \frac{1}{2}, 0)$, and its radius is $\sqrt{2}/2$. ■

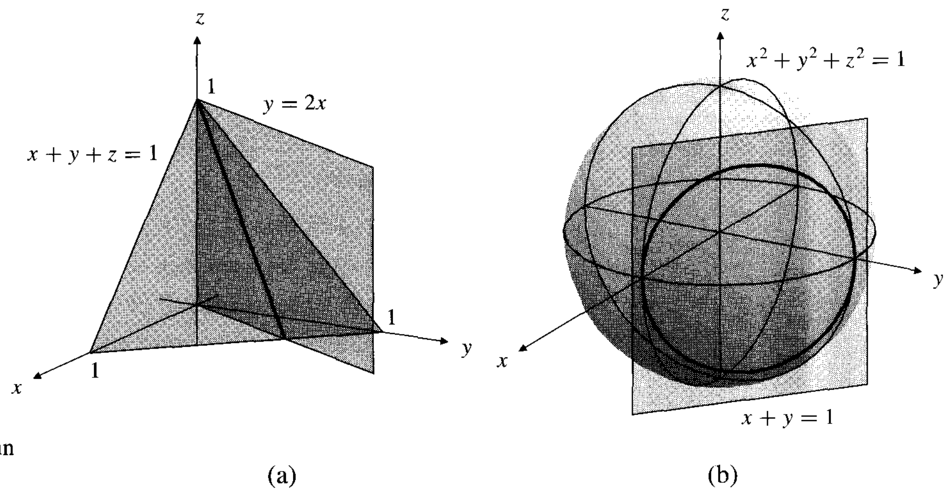


Figure 10.9
(a) The two planes intersect in a straight line
(b) The plane intersects the sphere in a circle

In Sections 10.4 and 10.5 we will see many more examples of geometric objects in 3-space represented by simple equations.

Euclidean *n*-Space

Mathematicians and users of mathematics frequently need to consider ***n*-dimensional space** where n is greater than 3, and may even be infinite. It is difficult to visualize a space of dimension 4 or higher geometrically. The secret to dealing with these spaces is to regard the points in n-space as *being* ordered n-tuples of real numbers; that is, $(x_1, x_2, \ldots, x_n)$ is a point in n-space instead of just being the coordinates of such a point. We stop thinking of points as existing in physical space and start thinking of them as algebraic objects. We usually denote n-space by the symbol $\mathbb{R}^n$ to show that its points are n-tuples of *real* numbers. Thus $\mathbb{R}^2$ and $\mathbb{R}^3$ denote the plane and 3-space, respectively. Note that in passing from $\mathbb{R}^3$ to $\mathbb{R}^n$ we have altered the notation a bit: in $\mathbb{R}^3$ we called the coordinates x, y, and z while in $\mathbb{R}^n$ we called them $x_1, x_2, \ldots$ and x_n so as not to run out of letters. We could, of course, talk about coordinates (x_1, x_2, x_3) in $\mathbb{R}^3$ and (x_1, x_2) in the plane $\mathbb{R}^2$, but (x, y, z) and (x, y) are traditionally used there.

Although we think of points in $\mathbb{R}^n$ as n-tuples rather than geometric objects, we do not want to lose all sight of the underlying geometry. By analogy with the two- and three-dimensional cases, we still consider the quantity

$$\sqrt{(y_1 - x_1)^2 + (y_2 - x_2)^2 + \cdots + (y_n - x_n)^2}$$

as representing the *distance* between the points with coordinates $(x_1, x_2, \ldots, x_n)$ and $(y_1, y_2, \ldots, y_n)$. Also, we call the $(n-1)$-dimensional set of points in $\mathbb{R}^n$ that satisfy the equation $x_n = 0$ a **hyperplane**, by analogy with the plane $z = 0$ in $\mathbb{R}^3$.

Describing Sets in the Plane, 3-Space, and *n*-Space

We conclude this section by collecting some definitions of terms used to describe sets of points in $\mathbb{R}^n$ for $n \geq 2$. These terms belong to the branch of mathematics called **topology**, and they generalize the notions of open and closed intervals and endpoints used to describe sets on the real line $\mathbb{R}$. We state the definitions for $\mathbb{R}^n$, but we are most interested in the cases where $n = 2$ or $n = 3$.

A **neighbourhood** of a point P in $\mathbb{R}^n$ is a set of the form

$$B_r(P) = \{Q \in \mathbb{R}^n : \text{distance from } Q \text{ to } P < r\}$$

for some $r > 0$.

For $n = 1$, if $p \in \mathbb{R}$, then $B_r(p)$ is the **open interval** $]p - r, p + r[$ centred at p.

For $n = 2$, $B_r(P)$ is the **open disk** of radius r centred at point P.

For $n = 3$, $B_r(P)$ is the **open ball** of radius r centred at point P.

A set S is **open** in $\mathbb{R}^n$ if every point of S has a neighbourhood contained in S. Every neighbourhood is itself an open set. Other examples of open sets in $\mathbb{R}^2$ include the sets of points (x, y) such that $x > 0$, or such that $y > x^2$, or even such that $y \neq x^2$. Typically, sets defined by strict inequalities (using "$>$" and "$<$") are open. Examples in $\mathbb{R}^3$ include the sets of points (x, y, z) satisfying $x + y + z > 2$, or $1 < x < 3$.

The whole space $\mathbb{R}^n$ is an open set in itself. For technical reasons, the empty set (containing no points) is also considered to be open. (No point in the empty set fails to have a neighbourhood contained in the empty set.)

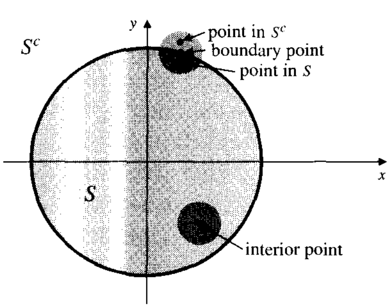


Figure 10.10 The closed disk S consisting of points $(x, y) \in \mathbb{R}^2$ satisfying $x^2 + y^2 \leq 1$. Note the shaded neighbourhoods of the boundary point and the interior point.
bdry(S) is the circle $x^2 + y^2 = 1$
int(S) is the open disk $x^2 + y^2 < 1$
ext(S) is the open set $x^2 + y^2 > 1$

The **complement**, S^c, of a set S in $\mathbb{R}^n$ is the set of all points in $\mathbb{R}^n$ that do not belong to S. For example, the complement of the set of points (x, y) in $\mathbb{R}^2$ such that $x > 0$ is the set of points for which $x \leq 0$. A set is said to be **closed** if its complement is open. Typically, sets defined by nonstrict inequalities (using "$\geq$" and "$\leq$") are closed. Closed intervals are closed sets in $\mathbb{R}$. Since the whole space and the empty set are both open in $\mathbb{R}^n$ and are complements of each other, they are also both closed. They are the only sets that are both open and closed.

A point P is called a **boundary point** of a set S if every neighbourhood of P contains both points in S and points in S^c. The **boundary**, bdry(S), of a set S is the set of all boundary points of S. For example, the boundary of the closed disk $x^2 + y^2 \leq 1$ in $\mathbb{R}^2$ is the circle $x^2 + y^2 = 1$. A closed set contains all its boundary points. An open set contains none of its boundary points.

A point P is an **interior point** of a set S if it belongs to S but not to the boundary of S. P is an **exterior point** of S if it belongs to the complement of S but not to the boundary of S. The **interior**, int(S), and **exterior**, ext(S), of S consist of all the interior points and exterior points of S, respectively. Both int(S) and ext(S) are open sets. If S is open, then int$(S) = S$. If S is closed, then ext$(S) = S^c$. See Figure 10.10.

Exercises 10.1

Find the distance between the pairs of points in Exercises 1–4.

1. $(0, 0, 0)$ and $(2, -1, -2)$

2. $(-1, -1, -1)$ and $(1, 1, 1)$

3. $(1, 1, 0)$ and $(0, 2, -2)$

4. $(3, 8, -1)$ and $(-2, 3, -6)$

5. What is the shortest distance from the point (x, y, z) to (a) the xy-plane? (b) the x-axis?

6. Show that the triangle with vertices $(1, 2, 3)$, $(4, 0, 5)$, and $(3, 6, 4)$ has a right angle.

7. Find the angle A in the triangle with vertices $A = (2, -1, -1)$, $B = (0, 1, -2)$, and $C = (1, -3, 1)$.

8. Show that the triangle with vertices $(1, 2, 3)$, $(1, 3, 4)$, and $(0, 3, 3)$ is equilateral.

9. Find the area of the triangle with vertices $(1, 1, 0)$, $(1, 0, 1)$, and $(0, 1, 1)$.

10. What is the distance from the origin to the point $(1, 1, \ldots, 1)$ in $\mathbb{R}^n$?

11. What is the distance from the point $(1, 1, \ldots, 1)$ in n-space to the closest point on the x_1-axis?

In Exercises 12–23, describe (and sketch if possible) the set of points in $\mathbb{R}^3$ that satisfy the given equation or inequality.

12. $z = 2$

13. $y \ge -1$

14. $z = x$

15. $x + y = 1$

16. $x^2 + y^2 + z^2 = 4$

17. $(x - 1)^2 + (y + 2)^2 + (z - 3)^2 = 4$

18. $x^2 + y^2 + z^2 = 2z$

19. $y^2 + z^2 \le 4$

20. $x^2 + z^2 = 4$

21. $z = y^2$

22. $z \ge \sqrt{x^2 + y^2}$

23. $x + 2y + 3z = 6$

In Exercises 24–32, describe (and sketch if possible) the set of points in $\mathbb{R}^3$ that satisfy the given pair of equations or inequalities.

24. $\begin{cases} x = 1 \\ y = 2 \end{cases}$

25. $\begin{cases} x = 1 \\ y = z \end{cases}$

26. $\begin{cases} x^2 + y^2 + z^2 = 4 \\ z = 1 \end{cases}$

27. $\begin{cases} x^2 + y^2 + z^2 = 4 \\ x^2 + y^2 + z^2 = 4x \end{cases}$

28. $\begin{cases} x^2 + y^2 + z^2 = 4 \\ x^2 + z^2 = 1 \end{cases}$

29. $\begin{cases} x^2 + y^2 = 1 \\ z = x \end{cases}$

30. $\begin{cases} y \ge x \\ z \le y \end{cases}$

31. $\begin{cases} x^2 + y^2 \le 1 \\ z \ge y \end{cases}$

32. $\begin{cases} x^2 + y^2 + z^2 \le 1 \\ \sqrt{x^2 + y^2} \le z \end{cases}$

In Exercises 33–36, specify the boundary and the interior of the plane sets S whose points (x, y) satisfy the given conditions. Is S open, closed, or neither?

33. $0 < x^2 + y^2 < 1$

34. $x \ge 0, \quad y < 0$

35. $x + y = 1$

36. $|x| + |y| \le 1$

In Exercises 37–40, specify the boundary and the interior of the sets S in 3-space whose points (x, y, z) satisfy the given conditions. Is S open, closed, or neither?

37. $1 \le x^2 + y^2 + z^2 \le 4$

38. $x \ge 0, \quad y > 1, \quad z < 2$

39. $(x - z)^2 + (y - z)^2 = 0$

40. $x^2 + y^2 < 1, \ y + z > 2$

10.2 Vectors

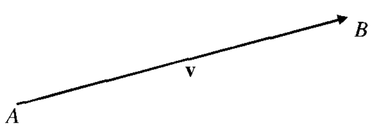


Figure 10.11 The vector $\mathbf{v} = \overrightarrow{AB}$

A **vector** is a quantity that involves both **magnitude** (size or length) and **direction**. For instance, the *velocity* of a moving object involves its speed and direction of motion, so is a vector. Such quantities are represented geometrically by arrows (directed line segments) and are often actually identified with these arrows. For instance, the vector $\overrightarrow{AB}$ is an arrow with tail at the point A and head at the point B. In print, such a vector is usually denoted by a single letter in boldface type,

$$\mathbf{v} = \overrightarrow{AB}.$$

(See Figure 10.11.) In handwriting, an arrow over a letter ($\vec{v} = \overrightarrow{AB}$) can be used to denote a vector. The *magnitude* of the vector $\mathbf{v}$ is the length of the arrow and is denoted $|\mathbf{v}|$ or $|\overrightarrow{AB}|$.

While vectors have magnitude and direction, they do not generally have *position*, that is, they are not regarded as being in a particular place. Two vectors, $\mathbf{u}$ and $\mathbf{v}$, are considered *equal* if they have *the same length and the same direction*, even

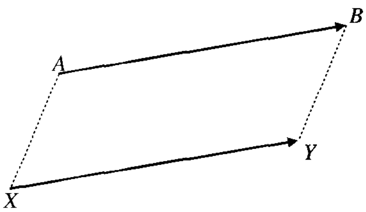


Figure 10.12 $\overrightarrow{AB} = \overrightarrow{XY}$

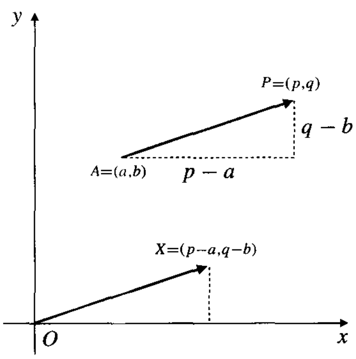


Figure 10.13 Components of a vector

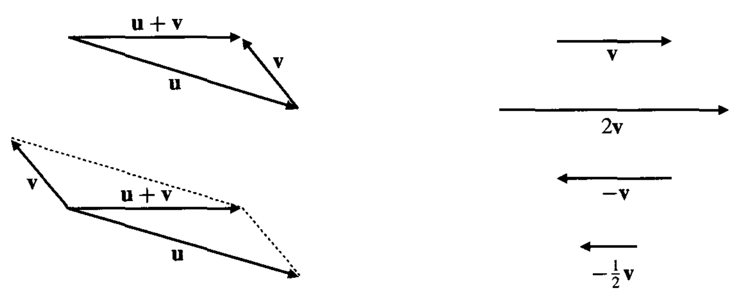
if their representative arrows do not coincide. The arrows must be parallel, have the same length, and point in the same direction. In Figure 10.12, for example, if $ABYX$ is a parallelogram, then $\overrightarrow{AB} = \overrightarrow{XY}$.

For the moment, we consider plane vectors, that is, vectors whose representative arrows lie in a plane. If we introduce a Cartesian coordinate system into the plane, we can talk about the x and y components of any vector. If $A = (a, b)$ and $P = (p, q)$, as shown in Figure 10.13, then the x and y components of $\overrightarrow{AP}$ are, respectively, $p - a$ and $q - b$. Note that if O is the origin and X is the point $(p - a, q - b)$, then

$$|\overrightarrow{AP}| = \sqrt{(p-a)^2 + (q-b)^2} = |\overrightarrow{OX}|$$

$$\text{slope of } \overrightarrow{AP} = \frac{q-b}{p-a} = \text{slope of } \overrightarrow{OX}.$$

Hence $\overrightarrow{AP} = \overrightarrow{OX}$. In general, two vectors are equal if and only if they have the same x components and y components.



(a) (b)

Figure 10.14
(a) Vector addition
(b) Scalar multiplication

There are two important algebraic operations defined for vectors: addition and scalar multiplication.

DEFINITION 1

Vector addition

Given two vectors **u** and **v**, their **sum u + v** is defined as follows. If an arrow representing **v** is placed with its tail at the head of an arrow representing **u**, then an arrow from the tail of **u** to the head of **v** represents **u** + **v**. Equivalently, if **u** and **v** have tails at the same point, then **u** + **v** is represented by an arrow with its tail at that point and its head at the opposite vertex of the parallelogram spanned by **u** and **v**. This is shown in Figure 10.14(a).

DEFINITION 2

Scalar multiplication

If **v** is a vector and t is a real number (also called a **scalar**), then the **scalar multiple** $t\mathbf{v}$ is a vector with magnitude $|t|$ times that of **v** and direction the same as **v** if $t > 0$, or opposite to that of **v** if $t < 0$. See Figure 10.14(b). If $t = 0$, then $t\mathbf{v}$ has zero length and therefore no particular direction. It is the **zero vector**, denoted **0**.

Suppose that **u** has components a and b and that **v** has components x and y. Then the components of $\mathbf{u}+\mathbf{v}$ are $a+x$ and $b+y$, and those of $t\mathbf{v}$ are tx and ty. See Figure 10.15.

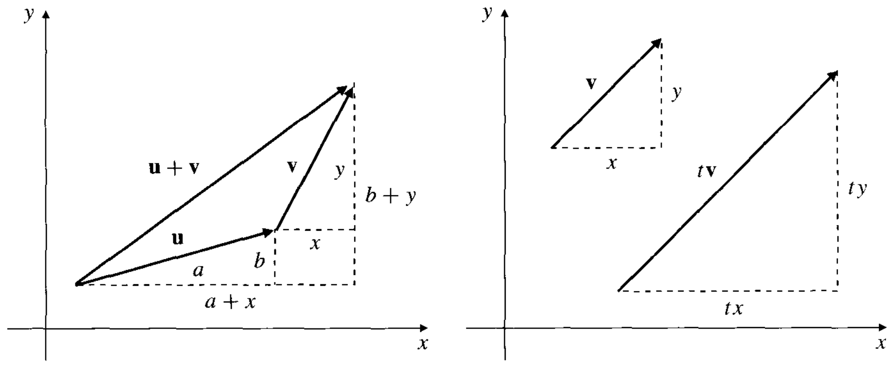


Figure 10.15 The components of a sum of vectors or a scalar multiple of a vector is the same sum or multiple of the corresponding components of the vectors

In $\mathbb{R}^2$ we single out two particular vectors for special attention. They are

(i) the vector **i** from the origin to the point $(1, 0)$, and

(ii) the vector **j** from the origin to the point $(0, 1)$.

Thus, **i** has components 1 and 0, and **j** has components 0 and 1. These vectors are called the **standard basis vectors** in the plane. The vector **r** from the origin to the point (x, y) has components x and y and can be expressed in the form

$$\mathbf{r} = \langle x, y \rangle = x\mathbf{i} + y\mathbf{j}.$$

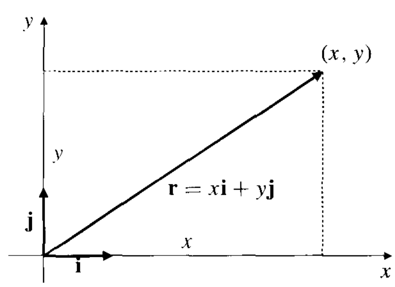


Figure 10.16 Any vector is a linear combination of the basis vectors

In the first form we specify the vector by listing its components between angle brackets; in the second we write **r** as a **linear combination** of the standard basis vectors **i** and **j**. (See Figure 10.16.) The vector **r** is called the **position vector** of the point (x, y). A position vector has its tail at the origin and its head at the point whose position it is specifying. The length of **r** is $|\mathbf{r}| = \sqrt{x^2 + y^2}$.

More generally, the vector $\overrightarrow{AP}$ from $A = (a, b)$ to $P = (p, q)$ in Figure 10.13 can also be written as a list of components or as a linear combination of the standard basis vectors:

$$\overrightarrow{AP} = \langle p - a, q - b \rangle = (p - a)\mathbf{i} + (q - b)\mathbf{j}.$$

Sums and scalar multiples of vectors are easily expressed in terms of components. If $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j}$ and $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j}$, and if t is a scalar (i.e., a real number), then

$$\mathbf{u} + \mathbf{v} = (u_1 + v_1)\mathbf{i} + (u_2 + v_2)\mathbf{j},$$
$$t\mathbf{u} = (tu_1)\mathbf{i} + (tu_2)\mathbf{j}.$$

The zero vector is $\mathbf{0} = 0\mathbf{i} + 0\mathbf{j}$. It has length zero and no specific direction. For any vector **u** we have $0\mathbf{u} = \mathbf{0}$. A **unit vector** is a vector of length 1. The standard basis vectors **i** and **j** are unit vectors. Given any nonzero vector **v**, we can form a unit vector $\hat{\mathbf{v}}$ in the same direction as **v** by multiplying **v** by the reciprocal of its length (a scalar):

$$\hat{\mathbf{v}} = \left(\frac{1}{|\mathbf{v}|}\right)\mathbf{v}.$$

Example 1 If $A = (2, -1)$, $B = (-1, 3)$, and $C = (0, 1)$, express each of the following vectors as a linear combination of the standard basis vectors:

(a) $\overrightarrow{AB}$ (b) $\overrightarrow{BC}$ (c) $\overrightarrow{AC}$ (d) $\overrightarrow{AB} + \overrightarrow{BC}$ (e) $2\overrightarrow{AC} - 3\overrightarrow{CB}$

(f) a unit vector in the direction of $\overrightarrow{AB}$.

Solution

(a) $\overrightarrow{AB} = (-1-2)\mathbf{i} + (3-(-1))\mathbf{j} = -3\mathbf{i} + 4\mathbf{j}$

(b) $\overrightarrow{BC} = (0-(-1))\mathbf{i} + (1-3)\mathbf{j} = \mathbf{i} - 2\mathbf{j}$

(c) $\overrightarrow{AC} = (0-2)\mathbf{i} + (1-(-1))\mathbf{j} = -2\mathbf{i} + 2\mathbf{j}$

(d) $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC} = -2\mathbf{i} + 2\mathbf{j}$

(e) $2\overrightarrow{AC} - 3\overrightarrow{CB} = 2(-2\mathbf{i} + 2\mathbf{j}) - 3(-\mathbf{i} + 2\mathbf{j}) = -\mathbf{i} - 2\mathbf{j}$

(f) A unit vector in the direction of $\overrightarrow{AB}$ is $\dfrac{\overrightarrow{AB}}{|\overrightarrow{AB}|} = -\dfrac{3}{5}\mathbf{i} + \dfrac{4}{5}\mathbf{j}$.

Implicit in the above example is the fact that the operations of addition and scalar multiplication obey appropriate algebraic rules, such as

$$\begin{aligned} \mathbf{u} + \mathbf{v} &= \mathbf{v} + \mathbf{u}, \\ (\mathbf{u} + \mathbf{v}) + \mathbf{w} &= \mathbf{u} + (\mathbf{v} + \mathbf{w}), \\ \mathbf{u} - \mathbf{v} &= \mathbf{u} + (-1)\mathbf{v}, \\ t(\mathbf{u} + \mathbf{v}) &= t\mathbf{u} + t\mathbf{v}. \end{aligned}$$

Vectors in 3-Space

The algebra and geometry of vectors described here extends to spaces of any number of dimensions; we can still think of vectors as represented by arrows, and sums and scalar multiples are formed just as for plane vectors.

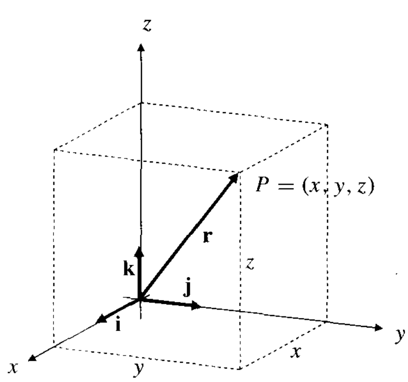


Figure 10.17 The standard basis vectors **i**, **j**, and **k**

Given a Cartesian coordinate system in 3-space, we define three **standard basis vectors**, **i**, **j**, and **k**, represented by arrows from the origin to the points $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$, respectively. (See Figure 10.17.) Any vector in 3-space can be written as a *linear combination* of these basis vectors; for instance, the position vector of the point (x, y, z) is given by

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}.$$

We say that **r** has **components** x, y, and z. The length of **r** is

$$|\mathbf{r}| = \sqrt{x^2 + y^2 + z^2}.$$

If $P_1 = (x_1, y_1, z_1)$ and $P_2 = (x_2, y_2, z_2)$ are two points in 3-space, then the vector $\mathbf{v} = \overrightarrow{P_1P_2}$ from P_1 to P_2 has components $x_2 - x_1$, $y_2 - y_1$, and $z_2 - z_1$ and is therefore represented in terms of the standard basis vectors by

$$\mathbf{v} = \overrightarrow{P_1P_2} = (x_2 - x_1)\mathbf{i} + (y_2 - y_1)\mathbf{j} + (z_2 - z_1)\mathbf{k}.$$

Example 2 If $\mathbf{u} = 2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$ and $\mathbf{v} = 3\mathbf{i} - 2\mathbf{j} - \mathbf{k}$, find $\mathbf{u} + \mathbf{v}$, $\mathbf{u} - \mathbf{v}$, $3\mathbf{u} - 2\mathbf{v}$, $|\mathbf{u}|$, $|\mathbf{v}|$, and a unit vector $\hat{\mathbf{u}}$ in the direction of $\mathbf{u}$.

Solution

$$\begin{aligned}
\mathbf{u} + \mathbf{v} &= (2+3)\mathbf{i} + (1-2)\mathbf{j} + (-2-1)\mathbf{k} = 5\mathbf{i} - \mathbf{j} - 3\mathbf{k} \\
\mathbf{u} - \mathbf{v} &= (2-3)\mathbf{i} + (1+2)\mathbf{j} + (-2+1)\mathbf{k} = -\mathbf{i} + 3\mathbf{j} - \mathbf{k} \\
3\mathbf{u} - 2\mathbf{v} &= (6-6)\mathbf{i} + (3+4)\mathbf{j} + (-6+2)\mathbf{k} = 7\mathbf{j} - 4\mathbf{k} \\
|\mathbf{u}| &= \sqrt{4+1+4} = 3, \qquad |\mathbf{v}| = \sqrt{9+4+1} = \sqrt{14} \\
\hat{\mathbf{u}} &= \left(\frac{1}{|\mathbf{u}|}\right)\mathbf{u} = \frac{2}{3}\mathbf{i} + \frac{1}{3}\mathbf{j} - \frac{2}{3}\mathbf{k}.
\end{aligned}$$

■

The following example illustrates the way vectors can be used to solve problems involving relative velocities. If A moves with velocity $\mathbf{v}_{A \text{ rel } B}$ relative to B, and B moves with velocity $\mathbf{v}_{B \text{ rel } C}$ relative to C, then A moves with velocity $\mathbf{v}_{A \text{ rel } C}$ relative to C, where

$$\mathbf{v}_{A \text{ rel } C} = \mathbf{v}_{A \text{ rel } B} + \mathbf{v}_{B \text{ rel } C}.$$

Example 3 An aircraft cruises at a speed of 300 km/h in still air. If the wind is blowing from the east at 100 km/h, in what direction should the aircraft head in order to fly in a straight line from city P to city Q, 400 km north northeast of P? How long will the trip take?

Solution The problem is two-dimensional, so we use plane vectors. Let us choose our coordinate system so that the x- and y-axes point east and north, respectively. Figure 10.18 illustrates the three velocities that must be considered. The velocity of the air relative to the ground is

$$\mathbf{v}_{\text{air rel ground}} = -100\,\mathbf{i}.$$

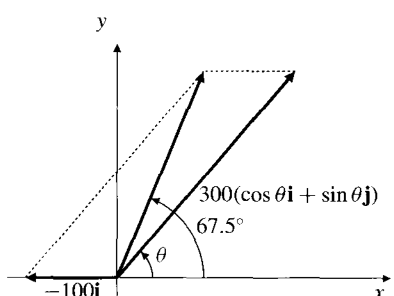


Figure 10.18 Velocity diagram for the aircraft in Example 3

If the aircraft heads in a direction making angle θ with the positive direction of the x-axis, then the velocity of the aircraft relative to the air is

$$\mathbf{v}_{\text{aircraft rel air}} = 300\cos\theta\,\mathbf{i} + 300\sin\theta\,\mathbf{j}.$$

Thus, the velocity of the aircraft relative to the ground is

$$\begin{aligned}
\mathbf{v}_{\text{aircraft rel ground}} &= \mathbf{v}_{\text{aircraft rel air}} + \mathbf{v}_{\text{air rel ground}} \\
&= (300\cos\theta - 100)\,\mathbf{i} + 300\sin\theta\,\mathbf{j}.
\end{aligned}$$

We want this latter velocity to be in a north-northeasterly direction, that is, in the direction making angle $3\pi/8 = 67.5°$ with the positive direction of the x-axis. Thus we will have

$$\mathbf{v}_{\text{aircraft rel ground}} = v\,[(\cos 67.5°)\ \mathbf{i} + (\sin 67.5°)\ \mathbf{j}],$$

where v is the actual groundspeed of the aircraft. Comparing the two expressions for $\mathbf{v}_{\text{aircraft rel ground}}$ we obtain

$$300\cos\theta - 100 = v\cos 67.5^\circ$$
$$300\sin\theta = v\sin 67.5^\circ.$$

Eliminating v between these two equations we get

$$300\cos\theta\sin 67.5^\circ - 300\sin\theta\cos 67.5^\circ = 100\sin 67.5^\circ,$$

or

$$3\sin(67.5^\circ - \theta) = \sin 67.5^\circ.$$

Therefore, the aircraft should head in direction θ given by

$$\theta = 67.5^\circ - \arcsin\left(\frac{1}{3}\sin 67.5^\circ\right) \approx 49.56^\circ,$$

that is, 49.56° north of east. The groundspeed is now seen to be

$$v = 300\sin\theta/\sin 67.5^\circ \approx 247.15 \text{ km/h}.$$

Thus, the 400 km trip will take about $400/247.15 \approx 1.618$ hours, or about 1 hour and 37 minutes. ■

Hanging Cables and Chains

When it is suspended from both ends and allowed to hang under gravity, a heavy cable or chain assumes the shape of a **catenary** curve, which is the graph of the hyperbolic cosine function. We will demonstrate this now, using vectors to keep track of the various forces acting on the cable.

Suppose that the cable has line density δ (units of mass per unit length) and hangs as shown in Figure 10.19. Let us choose a coordinate system so that the lowest point L on the cable is at $(0, y_0)$; we will specify the value of y_0 later. If $P = (x, y)$ is another point on the cable, there are three forces acting on the arc LP of the cable between L and P. These are all forces that we can represent using horizontal and vertical components.

(i) The horizontal tension $\mathbf{H} = -H\mathbf{i}$ at L. This is the force that the part of the cable to the left of L exerts on the arc LP at L.

(ii) The tangential tension $\mathbf{T} = T_h\mathbf{i} + T_v\mathbf{j}$. This is the force the part of the cable to the right of P exerts on arc LP at P.

(iii) The weight $\mathbf{W} = -\delta g s\mathbf{j}$ of arc LP, where g is the acceleration of gravity and s is the length of the arc LP.

Since the cable is not moving, these three forces must balance; their vector sum must be zero:

$$\mathbf{T} + \mathbf{H} + \mathbf{W} = \mathbf{0}$$
$$(T_h - H)\mathbf{i} + (T_v - \delta g s)\mathbf{j} = \mathbf{0}$$

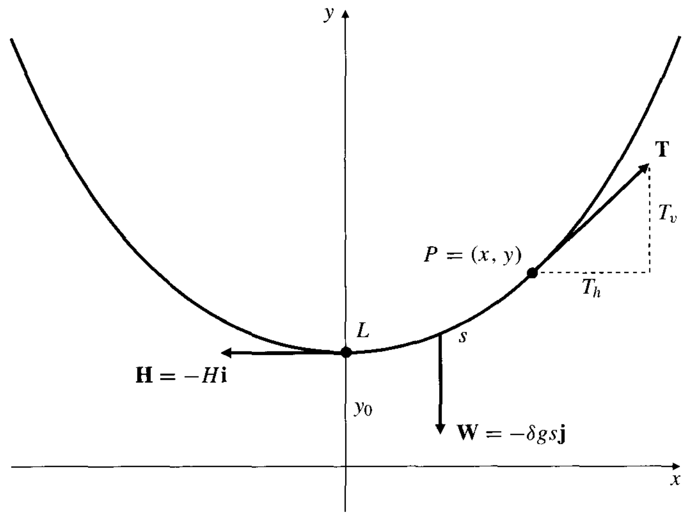


Figure 10.19 A hanging cable and the forces acting on arc LP

Thus $T_h = H$ and $T_v = \delta g s$. Since **T** is tangent to the cable at P, the slope of the cable there is

$$\frac{dy}{dx} = \frac{T_v}{T_h} = \frac{\delta g s}{H} = as,$$

where $a = \delta g/H$ is a constant for the given cable. Differentiating with respect to x and using the fact, from our study of arc length, that

$$\frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2},$$

we obtain a second-order differential equation,

$$\frac{d^2y}{dx^2} = a\frac{ds}{dx} = a\sqrt{1 + \left(\frac{dy}{dx}\right)^2},$$

to be solved for the equation of the curve along which the hanging cable lies. The appropriate initial conditions are $y = y_0$ and $dy/dx = 0$ at $x = 0$.

Since the differential equation depends on dy/dx rather than y, we substitute $m(x) = dy/dx$ and obtain a first-order equation for m:

$$\frac{dm}{dx} = a\sqrt{1 + m^2}.$$

This equation is separable; we integrate it using the substitution $m = \sinh u$:

$$\int \frac{1}{\sqrt{1+m^2}}\,dm = \int a\,dx$$

$$\int du = \int \frac{\cosh u}{\sqrt{1 + \sinh^2 u}}\,du = ax + C_1$$

$$\sinh^{-1} m = u = ax + C_1$$

$$m = \sinh(ax + C_1).$$

Since $m = dy/dx = 0$ at $x = 0$, we have $0 = \sinh C_1$, so $C_1 = 0$ and

$$\frac{dy}{dx} = m = \sinh(ax).$$

This equation is easily integrated to find y. (Had we used a tangent substitution instead of the hyperbolic sine substitution for m we would have had more trouble here.)

$$y = \frac{1}{a}\cosh(ax) + C_2.$$

If we choose $y_0 = y(0) = 1/a$, then, substituting $x = 0$ we will get $C_2 = 0$. With this choice of y_0, we therefore find that the equation of the curve along which the hanging cable lies is the catenary

$$y = \frac{1}{a}\cosh(ax).$$

Remark If a hanging cable bears loads other than its own weight, it will assume a different shape. For example, a cable supporting a level suspension bridge whose weight per unit length is much greater than that of the cable will assume the shape of a parabola. See Exercise 34 below.

The Dot Product and Projections

There is another operation on vectors in any dimension by which two vectors are combined to produce a number called their *dot product.*

DEFINITION 3

The dot product of two vectors

Given two vectors, $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j}$ and $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j}$ in $\mathbb{R}^2$, we define their **dot product** $\mathbf{u} \bullet \mathbf{v}$ to be the sum of the products of their corresponding components:

$$\mathbf{u} \bullet \mathbf{v} = u_1v_1 + u_2v_2.$$

The terms **scalar product** and **inner product** are also used in place of dot product. Similarly, for vectors $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$ and $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ in $\mathbb{R}^3$,

$$\mathbf{u} \bullet \mathbf{v} = u_1v_1 + u_2v_2 + u_3v_3.$$

The dot product has the following algebraic properties, easily checked using the definition above:

$$\begin{aligned}
&\mathbf{u} \bullet \mathbf{v} = \mathbf{v} \bullet \mathbf{u} && \text{(commutative law)},\\
&\mathbf{u} \bullet (\mathbf{v} + \mathbf{w}) = \mathbf{u} \bullet \mathbf{v} + \mathbf{u} \bullet \mathbf{w} && \text{(distributive law)},\\
&(t\mathbf{u}) \bullet \mathbf{v} = \mathbf{u} \bullet (t\mathbf{v}) = t(\mathbf{u} \bullet \mathbf{v}) && \text{(for real } t),\\
&\mathbf{u} \bullet \mathbf{u} = |\mathbf{u}|^2.
\end{aligned}$$

The real significance of the dot product is shown by the following result, which could have been used as the definition of dot product:

THEOREM 1

If θ is the angle between the directions of $\mathbf{u}$ and $\mathbf{v}$ $(0 \le \theta \le \pi)$, then

$$\mathbf{u} \bullet \mathbf{v} = |\mathbf{u}||\mathbf{v}| \cos\theta.$$

In particular, $\mathbf{u} \bullet \mathbf{v} = 0$ if and only if $\mathbf{u}$ and $\mathbf{v}$ are perpendicular. (Of course, the zero vector is perpendicular to every vector.)

PROOF Refer to Figure 10.20 and apply the Cosine Law to the triangle with the arrows $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{u} - \mathbf{v}$ as sides.

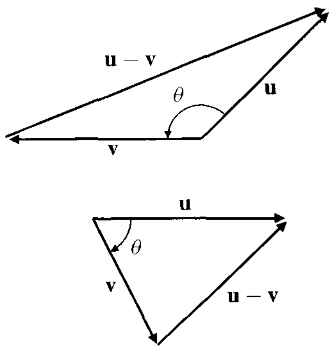


Figure 10.20

$$\begin{aligned} |\mathbf{u}|^2 + |\mathbf{v}|^2 - 2|\mathbf{u}|\,|\mathbf{v}| \cos\theta &= |\mathbf{u} - \mathbf{v}|^2 = (\mathbf{u} - \mathbf{v}) \bullet (\mathbf{u} - \mathbf{v}) \\ &= \mathbf{u} \bullet (\mathbf{u} - \mathbf{v}) - \mathbf{v} \bullet (\mathbf{u} - \mathbf{v}) \\ &= \mathbf{u} \bullet \mathbf{u} - \mathbf{u} \bullet \mathbf{v} - \mathbf{v} \bullet \mathbf{u} + \mathbf{v} \bullet \mathbf{v} \\ &= |\mathbf{u}|^2 + |\mathbf{v}|^2 - 2\mathbf{u} \bullet \mathbf{v} \end{aligned}$$

Hence $|\mathbf{u}||\mathbf{v}| \cos\theta = \mathbf{u} \bullet \mathbf{v}$, as claimed.

Example 4 Find the angle θ between the vectors $\mathbf{u} = 2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$ and $\mathbf{v} = 3\mathbf{i} - 2\mathbf{j} - \mathbf{k}$.

Solution Solving the formula $\mathbf{u} \bullet \mathbf{v} = |\mathbf{u}||\mathbf{v}| \cos\theta$ for θ, we obtain

$$\begin{aligned} \theta = \cos^{-1} \frac{\mathbf{u} \bullet \mathbf{v}}{|\mathbf{u}||\mathbf{v}|} &= \cos^{-1} \left(\frac{(2)(3) + (1)(-2) + (-2)(-1)}{3\sqrt{14}} \right) \\ &= \cos^{-1} \frac{2}{\sqrt{14}} \approx 57.69^\circ. \end{aligned}$$

It is sometimes useful to project one vector along another. We define both scalar and vector projections of $\mathbf{u}$ in the direction of $\mathbf{v}$:

DEFINITION 4

Scalar and vector projections

The **scalar projection** s of any vector $\mathbf{u}$ in the direction of a nonzero vector $\mathbf{v}$ is the dot product of $\mathbf{u}$ with a unit vector in the direction of $\mathbf{v}$. Thus, it is the *number*

$$s = \frac{\mathbf{u} \bullet \mathbf{v}}{|\mathbf{v}|} = |\mathbf{u}| \cos\theta,$$

where θ is the angle between $\mathbf{u}$ and $\mathbf{v}$.

The **vector projection**, $\mathbf{u}_\mathbf{v}$, of $\mathbf{u}$ in the direction of $\mathbf{v}$ (see Figure 10.21) is the scalar multiple of a unit vector $\hat{\mathbf{v}}$ in the direction of $\mathbf{v}$, by the scalar projection of $\mathbf{u}$ in the direction of $\mathbf{v}$, that is,

$$\text{vector projection of } \mathbf{u} \text{ along } \mathbf{v} = \mathbf{u}_\mathbf{v} = \frac{\mathbf{u} \bullet \mathbf{v}}{|\mathbf{v}|} \hat{\mathbf{v}} = \frac{\mathbf{u} \bullet \mathbf{v}}{|\mathbf{v}|^2} \mathbf{v}.$$

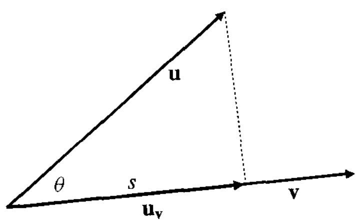


Figure 10.21 The scalar projection s and the vector projection $\mathbf{u_v}$ of vector $\mathbf{u}$ along vector $\mathbf{v}$

Note that $|s|$ is the length of the line segment along the line of $\mathbf{v}$ obtained by dropping perpendiculars to that line from the tail and head of $\mathbf{u}$. (See Figure 10.21.) Also, s is negative if $\theta > 90°$.

It is often necessary to express a vector as a sum of two other vectors parallel and perpendicular to a given direction.

Example 5 Express the vector $3\mathbf{i} + \mathbf{j}$ as a sum of vectors $\mathbf{u} + \mathbf{v}$, where $\mathbf{u}$ is parallel to the vector $\mathbf{i} + \mathbf{j}$ and $\mathbf{v}$ is perpendicular to $\mathbf{u}$.

Solution

Method I (Using vector projection) Note that $\mathbf{u}$ must be the vector projection of $3\mathbf{i} + \mathbf{j}$ in the direction of $\mathbf{i} + \mathbf{j}$. Thus,

$$\mathbf{u} = \frac{(3\mathbf{i} + \mathbf{j}) \bullet (\mathbf{i} + \mathbf{j})}{|\mathbf{i} + \mathbf{j}|^2}(\mathbf{i} + \mathbf{j}) = \frac{4}{2}(\mathbf{i} + \mathbf{j}) = 2\mathbf{i} + 2\mathbf{j}$$
$$\mathbf{v} = 3\mathbf{i} + \mathbf{j} - \mathbf{u} = \mathbf{i} - \mathbf{j}.$$

Method II (From basic principles) Since $\mathbf{u}$ is parallel to $\mathbf{i}+\mathbf{j}$ and $\mathbf{v}$ is perpendicular to $\mathbf{u}$, we have

$$\mathbf{u} = t(\mathbf{i} + \mathbf{j}) \qquad \text{and} \qquad \mathbf{v} \bullet (\mathbf{i} + \mathbf{j}) = 0,$$

for some scalar t. We want $\mathbf{u} + \mathbf{v} = 3\mathbf{i} + \mathbf{j}$. Take the dot product of this equation with $\mathbf{i} + \mathbf{j}$:

$$\mathbf{u} \bullet (\mathbf{i} + \mathbf{j}) + \mathbf{v} \bullet (\mathbf{i} + \mathbf{j}) = (3\mathbf{i} + \mathbf{j}) \bullet (\mathbf{i} + \mathbf{j})$$
$$t(\mathbf{i} + \mathbf{j}) \bullet (\mathbf{i} + \mathbf{j}) + 0 = 4.$$

Thus $2t = 4$, so $t = 2$. Therefore,

$$\mathbf{u} = 2\mathbf{i} + 2\mathbf{j} \qquad \text{and} \qquad \mathbf{v} = 3\mathbf{i} + \mathbf{j} - \mathbf{u} = \mathbf{i} - \mathbf{j}.$$

Vectors in *n*-Space

All the above ideas make sense for vectors in spaces of any dimension. Vectors in $\mathbb{R}^n$ can be expressed as linear combinations of the n unit vectors

$\mathbf{e_1}$	from the origin to the point	$(1, 0, 0, \ldots, 0)$
$\mathbf{e_2}$	from the origin to the point	$(0, 1, 0, \ldots, 0)$
	$\vdots$	
$\mathbf{e_n}$	from the origin to the point	$(0, 0, 0, \ldots, 1)$.

These vectors constitute a *standard basis* in $\mathbb{R}^n$. The n-vector $\mathbf{x}$ with components $x_1, x_2, \ldots, x_n$ is expressed in the form

$$\mathbf{x} = x_1\mathbf{e_1} + x_2\mathbf{e_2} + \cdots + x_n\mathbf{e_n}.$$

The length of $\mathbf{x}$ is $|\mathbf{x}| = \sqrt{x_1{}^2 + x_2{}^2 + \cdots + x_n{}^2}$. The angle between two vectors $\mathbf{x}$ and $\mathbf{y}$ is

$$\theta = \cos^{-1} \frac{\mathbf{x} \bullet \mathbf{y}}{|\mathbf{x}||\mathbf{y}|},$$

where

$$\mathbf{x} \bullet \mathbf{y} = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n.$$

We will not make much use of n-vectors for $n > 3$ but you should be aware that everything said up until now for 2-vectors or 3-vectors extends to n-vectors.

Exercises 10.2

1. Let $A = (-1, 2)$, $B = (2, 0)$, $C = (1, -3)$, $D = (0, 4)$. Express each of the following vectors as a linear combination of the standard basis vectors $\mathbf{i}$ and $\mathbf{j}$ in $\mathbb{R}^2$.

   (a) $\overrightarrow{AB}$, (b) $\overrightarrow{BA}$, (c) $\overrightarrow{AC}$, (d) $\overrightarrow{BD}$, (e) $\overrightarrow{DA}$,

   (f) $\overrightarrow{AB} - \overrightarrow{BC}$, (g) $\overrightarrow{AC} - 2\overrightarrow{AB} + 3\overrightarrow{CD}$,

   (h) $\dfrac{\overrightarrow{AB} + \overrightarrow{AC} + \overrightarrow{AD}}{3}$.

In Exercises 2–3, calculate the following for the given vectors $\mathbf{u}$ and $\mathbf{v}$:

(a) $\mathbf{u} + \mathbf{v}$, $\mathbf{u} - \mathbf{v}$, $2\mathbf{u} - 3\mathbf{v}$,

(b) the lengths $|\mathbf{u}|$ and $|\mathbf{v}|$,

(c) unit vectors $\hat{\mathbf{u}}$ and $\hat{\mathbf{v}}$ in the directions of $\mathbf{u}$ and $\mathbf{v}$, respectively,

(d) the dot product $\mathbf{u} \bullet \mathbf{v}$,

(e) the angle between $\mathbf{u}$ and $\mathbf{v}$,

(f) the scalar projection of $\mathbf{u}$ in the direction of $\mathbf{v}$,

g) the vector projection of $\mathbf{v}$ along $\mathbf{u}$.

2. $\mathbf{u} = \mathbf{i} - \mathbf{j}$ and $\mathbf{v} = \mathbf{j} + 2\mathbf{k}$
3. $\mathbf{u} = 3\mathbf{i} + 4\mathbf{j} - 5\mathbf{k}$ and $\mathbf{v} = 3\mathbf{i} - 4\mathbf{j} - 5\mathbf{k}$
4. Use vectors to show that the triangle with vertices $(-1, 1)$, $(2, 5)$, and $(10, -1)$ is right-angled.

In Exercises 5–8, prove the stated geometric result using vectors.

5. The line segment joining the midpoints of two sides of a triangle is parallel to and half as long as the third side.
6. If P, Q, R, and S are midpoints of sides AB, BC, CD, and DA, respectively, of quadrilateral $ABCD$, then $PQRS$ is a parallelogram.

* 7. The diagonals of any parallelogram bisect each other.

* 8. The medians of any triangle meet in a common point. (A median is a line joining one vertex to the midpoint of the opposite side. The common point is the *centroid* of the triangle.)

9. A weather vane mounted on the top of a car moving due north at 50 km/h indicates that the wind is coming from the west. When the car doubles its speed, the weather vane indicates that the wind is coming from the northwest. From what direction is the wind coming, and what is its speed?
10. A straight river 500 m wide flows due east at a constant speed of 3 km/h. If you can row your boat at a speed of 5 km/h in still water, in what direction should you head if you wish to row from point A on the south shore to point B on the north shore directly north of A? How long will the trip take?

* 11. In what direction should you head to cross the river in Exercise 10 if you can only row at 2 km/h, and you wish to row from A to point C on the north shore, k km downstream from B. For what values of k is the trip not possible?

12. A certain aircraft flies with an airspeed of 750 km/h. In what direction should it head in order to make progress in a true easterly direction if the wind is from the northeast at 100 km/h? How long will it take to complete a trip to a city 1,500 km from its starting point?
13. For what value of t is the vector $2t\mathbf{i} + 4\mathbf{j} - (10 + t)\mathbf{k}$ perpendicular to the vector $\mathbf{i} + t\mathbf{j} + \mathbf{k}$?
14. Find the angle between a diagonal of a cube and one of the edges of the cube.
15. Find the angle between a diagonal of a cube and a diagonal of one of the faces of the cube. Give all possible answers.
16. **(Direction cosines)** If a vector $\mathbf{u}$ in $\mathbb{R}^3$ makes angles α, β, and γ with the coordinate axes, show that

$$\hat{\mathbf{u}} = \cos\alpha\mathbf{i} + \cos\beta\mathbf{j} + \cos\gamma\mathbf{k}$$

is a unit vector in the direction of $\mathbf{u}$, so $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$. The numbers $\cos\alpha$, $\cos\beta$, and $\cos\gamma$ are called the *direction cosines* of $\mathbf{u}$.

17. Find a unit vector that makes equal angles with the three coordinate axes.

18. Find the three angles of the triangle with vertices $(1, 0, 0)$, $(0, 2, 0)$, and $(0, 0, 3)$.

19. If $\mathbf{r}_1$ and $\mathbf{r}_2$ are the position vectors of two points, P_1 and P_2, and λ is a real number, show that

$$\mathbf{r} = (1 - \lambda)\mathbf{r}_1 + \lambda\mathbf{r}_2$$

is the position vector of a point P on the straight line joining P_1 and P_2. Where is P if $\lambda = 1/2$? if $\lambda = 2/3$? if $\lambda = -1$? if $\lambda = 2$?

20. Let $\mathbf{a}$ be a nonzero vector. Describe the set of all points in 3-space whose position vectors $\mathbf{r}$ satisfy $\mathbf{a} \bullet \mathbf{r} = 0$.

21. Let $\mathbf{a}$ be a nonzero vector, and let b be any real number. Describe the set of all points in 3-space whose position vectors $\mathbf{r}$ satisfy $\mathbf{a} \bullet \mathbf{r} = b$.

In Exercises 22–24, $\mathbf{u} = 2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$, $\mathbf{v} = \mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$, and $\mathbf{w} = 2\mathbf{i} - 2\mathbf{j} + \mathbf{k}$.

22. Find two unit vectors each of which is perpendicular to $\mathbf{u}$ and $\mathbf{v}$.

23. Find a vector $\mathbf{x}$ satisfying the system of equations $\mathbf{x} \bullet \mathbf{u} = 9$, $\mathbf{x} \bullet \mathbf{v} = 4$, $\mathbf{x} \bullet \mathbf{w} = 6$.

24. Find two unit vectors each of which makes equal angles with $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$.

25. Find a unit vector that bisects the angle between any two nonzero vectors $\mathbf{u}$ and $\mathbf{v}$.

26. Given two nonparallel vectors $\mathbf{u}$ and $\mathbf{v}$, describe the set of all points whose position vectors $\mathbf{r}$ are of the form $\mathbf{r} = \lambda\mathbf{u} + \mu\mathbf{v}$, where λ and μ are arbitrary real numbers.

27. (The triangle inequality) Let $\mathbf{u}$ and $\mathbf{v}$ be two vectors.

(a) Show that $|\mathbf{u} + \mathbf{v}|^2 = |\mathbf{u}|^2 + 2\mathbf{u} \bullet \mathbf{v} + |\mathbf{v}|^2$.

(b) Show that $\mathbf{u} \bullet \mathbf{v} \le |\mathbf{u}||\mathbf{v}|$.

(c) Deduce from (a) and (b) that $|\mathbf{u} + \mathbf{v}| \le |\mathbf{u}| + |\mathbf{v}|$.

28. (a) Why is the inequality in Exercise 27(c) called a triangle inequality?

(b) What conditions on $\mathbf{u}$ and $\mathbf{v}$ imply that $|\mathbf{u} + \mathbf{v}| = |\mathbf{u}| + |\mathbf{v}|$?

29. (Orthonormal bases) Let $\mathbf{u} = \frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{j}$, $\mathbf{v} = \frac{4}{5}\mathbf{i} - \frac{3}{5}\mathbf{j}$, and $\mathbf{w} = \mathbf{k}$.

(a) Show that $|\mathbf{u}| = |\mathbf{v}| = |\mathbf{w}| = 1$ and $\mathbf{u} \bullet \mathbf{v} = \mathbf{u} \bullet \mathbf{w} = \mathbf{v} \bullet \mathbf{w} = 0$. The vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are mutually perpendicular unit vectors and as such are said to constitute an **orthonormal basis** for $\mathbb{R}^3$.

(b) If $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, show by direct calculation that

$$\mathbf{r} = (\mathbf{r} \bullet \mathbf{u})\mathbf{u} + (\mathbf{r} \bullet \mathbf{v})\mathbf{v} + (\mathbf{r} \bullet \mathbf{w})\mathbf{w}.$$

30. Show that if $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are any three mutually perpendicular unit vectors in $\mathbb{R}^3$ and $\mathbf{r} = a\mathbf{u} + b\mathbf{v} + c\mathbf{w}$, then $a = \mathbf{r} \bullet \mathbf{u}$, $b = \mathbf{r} \bullet \mathbf{v}$, and $c = \mathbf{r} \bullet \mathbf{w}$.

31. (Resolving a vector in perpendicular directions) If $\mathbf{a}$ is a nonzero vector and $\mathbf{w}$ is any vector, find vectors $\mathbf{u}$ and $\mathbf{v}$ such that $\mathbf{w} = \mathbf{u} + \mathbf{v}$, $\mathbf{u}$ is parallel to $\mathbf{a}$, and $\mathbf{v}$ is perpendicular to $\mathbf{a}$.

32. (Expressing a vector as a linear combination of two other vectors with which it is coplanar) Suppose that $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{r}$ are position vectors of points U, V, and P, respectively, that $\mathbf{u}$ is not parallel to $\mathbf{v}$, and that P lies in the plane containing the origin, U and V. Show that there exist numbers λ and μ such that $\mathbf{r} = \lambda\mathbf{u} + \mu\mathbf{v}$. *Hint:* resolve both $\mathbf{v}$ and $\mathbf{r}$ as sums of vectors parallel and perpendicular to $\mathbf{u}$ as suggested in Exercise 31.

* **33.** Given constants r, s, and t, with $r \ne 0$ and $s \ne 0$, and given a vector $\mathbf{a}$ satisfying $|\mathbf{a}|^2 > 4rst$, solve the system of equations

$$\begin{cases} r\mathbf{x} + s\mathbf{y} = \mathbf{a} \\ \mathbf{x} \bullet \mathbf{y} = t \end{cases}$$

for the unknown vectors $\mathbf{x}$ and $\mathbf{y}$.

Hanging cables

34. (A suspension bridge) If a hanging cable is supporting weight with constant horizontal line density (so that the weight supported by the arc LP in Figure 10.19 is $\delta g x$ rather than $\delta g s$, show that the cable assumes the shape of a parabola rather than a catenary. Such is likely to be the case for the cables of a suspension bridge.

35. At a point P, 10 m away horizontally from its lowest point L, a cable makes an angle 55° with the horizontal. Find the length of the cable between L and P.

36. Calculate the length s of the arc LP of the hanging cable in Figure 10.19 using the equation $y = (1/a)\cosh(ax)$ obtained for the cable. Hence, verify that the magnitude $T = |\mathbf{T}|$ of the tension in the cable at any point $P = (x, y)$ is $T = \delta g y$.

37. A cable 100 m long hangs between two towers 90 m apart so that its ends are attached at the same height on the two towers. How far below that height is the lowest point on the cable?

10.3 The Cross Product in 3-Space

There is defined, *in 3-space only*, another kind of product of two vectors called a *cross product* or *vector product*, and denoted $\mathbf{u}\times\mathbf{v}$.

DEFINITION 5

For any vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^3$, the **cross product** $\mathbf{u}\times\mathbf{v}$ is the unique vector satisfying the three conditions:

(i) $(\mathbf{u}\times\mathbf{v})\bullet\mathbf{u}=0$ and $(\mathbf{u}\times\mathbf{v})\bullet\mathbf{v}=0$,

(ii) $|\mathbf{u}\times\mathbf{v}|=|\mathbf{u}||\mathbf{v}|\sin\theta$, where θ is the angle between $\mathbf{u}$ and $\mathbf{v}$, and

(iii) $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{u}\times\mathbf{v}$ form a right-handed triad.

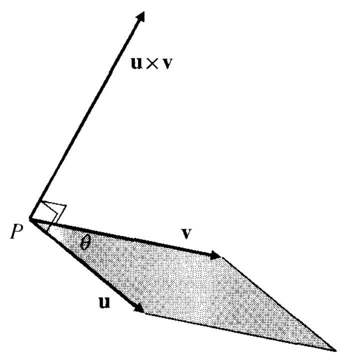


Figure 10.22 $\mathbf{u}\times\mathbf{v}$ is perpendicular to both $\mathbf{u}$ and $\mathbf{v}$ and has length equal to the area of the shaded parallelogram

If $\mathbf{u}$ and $\mathbf{v}$ are parallel, condition (ii) says that $\mathbf{u}\times\mathbf{v}=\mathbf{0}$, the zero vector. Otherwise, through any point in $\mathbb{R}^3$ there is a unique straight line that is perpendicular to both $\mathbf{u}$ and $\mathbf{v}$. Condition (i) says that $\mathbf{u}\times\mathbf{v}$ is parallel to this line. Condition (iii) determines which of the two directions along this line is the direction of $\mathbf{u}\times\mathbf{v}$; a right-handed screw advances in the direction of $\mathbf{u}\times\mathbf{v}$ if rotated in the direction from $\mathbf{u}$ toward $\mathbf{v}$. (This is equivalent to saying that the thumb, forefinger, and middle finger of the right hand can be made to point in the directions of $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{u}\times\mathbf{v}$, respectively.)

If $\mathbf{u}$ and $\mathbf{v}$ have their tails at the point P, then $\mathbf{u}\times\mathbf{v}$ is normal (i.e., perpendicular) to the plane through P in which $\mathbf{u}$ and $\mathbf{v}$ lie and, by condition (ii), $\mathbf{u}\times\mathbf{v}$ has length equal to the area of the parallelogram spanned by $\mathbf{u}$ and $\mathbf{v}$. (See Figure 10.22.) These properties make the cross product very useful for the description of tangent planes and normal lines to surfaces in $\mathbb{R}^3$.

The definition of cross product given above does not involve any coordinate system and therefore does not directly show the components of the cross product with respect to the standard basis. These components are provided by the following theorem:

THEOREM 2

Components of the cross product

If $\mathbf{u}=u_1\mathbf{i}+u_2\mathbf{j}+u_3\mathbf{k}$ and $\mathbf{v}=v_1\mathbf{i}+v_2\mathbf{j}+v_3\mathbf{k}$, then

$$\mathbf{u}\times\mathbf{v}=(u_2v_3-u_3v_2)\mathbf{i}+(u_3v_1-u_1v_3)\mathbf{j}+(u_1v_2-u_2v_1)\mathbf{k}.$$

PROOF First, we observe that the vector

$$\mathbf{w}=(u_2v_3-u_3v_2)\mathbf{i}+(u_3v_1-u_1v_3)\mathbf{j}+(u_1v_2-u_2v_1)\mathbf{k}$$

is perpendicular to both $\mathbf{u}$ and $\mathbf{v}$ since

$$\mathbf{u}\bullet\mathbf{w}=u_1(u_2v_3-u_3v_2)+u_2(u_3v_1-u_1v_3)+u_3(u_1v_2-u_2v_1)=0,$$

and similarly $\mathbf{v}\bullet\mathbf{w}=0$. Thus $\mathbf{u}\times\mathbf{v}$ is parallel to $\mathbf{w}$. Next, we show that $\mathbf{w}$ and $\mathbf{u}\times\mathbf{v}$ have the same length. In fact,

$$\begin{aligned}|\mathbf{w}|^2&=(u_2v_3-u_3v_2)^2+(u_3v_1-u_1v_3)^2+(u_1v_2-u_2v_1)^2\\&=u_2^2v_3^2+u_3^2v_2^2-2u_2v_3u_3v_2+u_3^2v_1^2+u_1^2v_3^2\\&\quad-2u_3v_1u_1v_3+u_1^2v_2^2+u_2^2v_1^2-2u_1v_2u_2v_1,\end{aligned}$$

while

$$\begin{aligned}|\mathbf{u}\times\mathbf{v}|^2 &= |\mathbf{u}|^2|\mathbf{v}|^2\sin^2\theta \\ &= |\mathbf{u}|^2|\mathbf{v}|^2(1-\cos^2\theta) \\ &= |\mathbf{u}|^2|\mathbf{v}|^2 - (\mathbf{u}\bullet\mathbf{v})^2 \\ &= (u_1^2+u_2^2+u_3^2)(v_1^2+v_2^2+v_3^2) - (u_1v_1+u_2v_2+u_3v_3)^2 \\ &= u_1^2v_1^2+u_1^2v_2^2+u_1^2v_3^2+u_2^2v_1^2+u_2^2v_2^2+u_2^2v_3^2+u_3^2v_1^2+u_3^2v_2^2+u_3^2v_3^2 \\ &\quad - u_1^2v_1^2 - u_2^2v_2^2 - u_3^2v_3^2 - 2u_1v_1u_2v_2 - 2u_1v_1u_3v_3 - 2u_2v_2u_3v_3 \\ &= |\mathbf{w}|^2.\end{aligned}$$

Since $\mathbf{w}$ is parallel to, and has the same length as, $\mathbf{u}\times\mathbf{v}$, we must have either $\mathbf{u}\times\mathbf{v}=\mathbf{w}$ or $\mathbf{u}\times\mathbf{v}=-\mathbf{w}$. It remains to be shown that the first of these is the correct choice. To see this, suppose that the triad of vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ is rigidly rotated in 3-space so that $\mathbf{u}$ points in the direction of the positive x-axis and $\mathbf{v}$ lies in the upper half of the xy-plane. Then $\mathbf{u}=u_1\mathbf{i}$, and $\mathbf{v}=v_1\mathbf{i}+v_2\mathbf{j}$, where $u_1>0$ and $v_2>0$. By the "right-hand rule" $\mathbf{u}\times\mathbf{v}$ must point in the direction of the positive z-axis. But $\mathbf{w}=u_1v_2\mathbf{k}$ does point in that direction, so $\mathbf{u}\times\mathbf{v}=\mathbf{w}$, as asserted.

The formula for the cross product in terms of components may seem awkward and asymmetric. As we shall see, however, it can be written more easily in terms of a determinant. We introduce determinants later in this section.

Example 1 **(Calculating cross products)**

(a)
$$\begin{aligned}&\mathbf{i}\times\mathbf{i}=\mathbf{0}, &&\mathbf{i}\times\mathbf{j}=\mathbf{k}, &&\mathbf{j}\times\mathbf{i}=-\mathbf{k},\\ &\mathbf{j}\times\mathbf{j}=\mathbf{0}, &&\mathbf{j}\times\mathbf{k}=\mathbf{i}, &&\mathbf{k}\times\mathbf{j}=-\mathbf{i},\\ &\mathbf{k}\times\mathbf{k}=\mathbf{0}, &&\mathbf{k}\times\mathbf{i}=\mathbf{j}, &&\mathbf{i}\times\mathbf{k}=-\mathbf{j}.\end{aligned}$$

(b) $(2\mathbf{i}+\mathbf{j}-3\mathbf{k})\times(-2\mathbf{j}+5\mathbf{k})$
$$\begin{aligned}&=\big((1)(5)-(-2)(-3)\big)\mathbf{i}+\big((-3)(0)-(2)(5)\big)\mathbf{j}+\big((2)(-2)-(1)(0)\big)\mathbf{k}\\ &=-\mathbf{i}-10\mathbf{j}-4\mathbf{k}.\end{aligned}$$

The cross product has some, but not all of the properties we usually ascribe to products. We summarize its algebraic properties as follows:

Properties of the cross product

If $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are any vectors in $\mathbb{R}^3$, and t is a real number (a scalar), then

(i) $\mathbf{u}\times\mathbf{u}=\mathbf{0}$,
(ii) $\mathbf{u}\times\mathbf{v}=-\mathbf{v}\times\mathbf{u}$, (The cross product is **anticommutative**.)
(iii) $(\mathbf{u}+\mathbf{v})\times\mathbf{w}=\mathbf{u}\times\mathbf{w}+\mathbf{v}\times\mathbf{w}$,
(iv) $\mathbf{u}\times(\mathbf{v}+\mathbf{w})=\mathbf{u}\times\mathbf{v}+\mathbf{u}\times\mathbf{w}$,
(v) $(t\mathbf{u})\times\mathbf{v}=\mathbf{u}\times(t\mathbf{v})=t(\mathbf{u}\times\mathbf{v})$,
(vi) $\mathbf{u}\bullet(\mathbf{u}\times\mathbf{v})=\mathbf{v}\bullet(\mathbf{u}\times\mathbf{v})=0$.

These identities are all easily verified using the components or the definition of the cross product or by using properties of determinants discussed below. They are left as exercises for the reader. Note the absence of an associative law. The cross product is not associative. (See Exercise 21 at the end of this section.) In general,

$$\mathbf{u}\times(\mathbf{v}\times\mathbf{w}) \neq (\mathbf{u}\times\mathbf{v})\times\mathbf{w}.$$

Determinants

In order to simplify certain formulas such as the component representation of the cross product, we introduce 2×2 and 3×3 **determinants**. General $n \times n$ determinants are normally studied in courses on linear algebra; we will encounter them in Section 10.6. In this section we will outline enough of the properties of determinants to enable us to use them as shorthand in some otherwise complicated formulas.

A determinant is an expression that involves the elements of a square array (matrix) of numbers. The determinant of the 2×2 array of numbers

$$\begin{matrix} a & b \\ c & d \end{matrix}$$

is denoted by enclosing the array between vertical bars, and its value is the number $ad - bc$:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

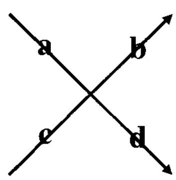


Figure 10.23

This is the product of elements in the *downward diagonal* of the array minus the product of elements in the *upward diagonal* as shown in Figure 10.23. For example,

$$\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = (1)(4) - (2)(3) = -2.$$

Similarly, the determinant of a 3×3 array of numbers is defined by

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = aei + bfg + cdh - gec - hfa - idb.$$

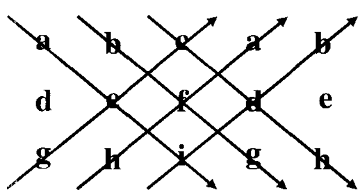


Figure 10.24 WARNING: This method does not work for 4×4 or higher-order determinants!

Observe that each of the six products in the value of the determinant involves exactly one element from each row and exactly one from each column of the array. As such, each term is the product of elements in a *diagonal* of an *extended* array obtained by repeating the first two columns of the array to the right of the third column, as shown in Figure 10.24. The value of the determinant is the sum of products corresponding to the three complete *downward* diagonals minus the sum corresponding to the three *upward* diagonals. With practice you will be able to form these diagonal products without having to write the extended array.

If we group the terms in the expansion of the determinant to factor out the elements of the first row, we obtain

$$\begin{aligned}\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} &= a(ei - fh) - b(di - fg) + c(dh - eg) \\ &= a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}.\end{aligned}$$

The 2×2 determinants appearing here (called *minors* of the given 3×3 determinant) are obtained by deleting the row and column containing the corresponding element from the original 3×3 determinant. This process is called *expanding* the 3×3 determinant *in minors* about the first row.

Such expansions in minors can be carried out about any row or column. Note that a minus sign appears in any term whose minor is obtained by deleting the ith row and jth column, where $i+j$ is an *odd* number. For example, we can expand the above determinant in minors about the second column as follows:

$$\begin{aligned}\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} &= -b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + e\begin{vmatrix} a & c \\ g & i \end{vmatrix} - h\begin{vmatrix} a & c \\ d & f \end{vmatrix} \\ &= -bdi + bfg + eai - ecg - haf + hcd.\end{aligned}$$

(Of course, this is the same value as the one obtained previously.)

Example 2

$$\begin{aligned}\begin{vmatrix} 1 & 4 & -2 \\ -3 & 1 & 0 \\ 2 & 2 & -3 \end{vmatrix} &= 3\begin{vmatrix} 4 & -2 \\ 2 & -3 \end{vmatrix} + 1\begin{vmatrix} 1 & -2 \\ 2 & -3 \end{vmatrix} \\ &= 3(-8) + 1 = -23.\end{aligned}$$

We expanded about the second row; the third column would also have been a good choice. (Why?) ■

Any row (or column) of a determinant may be regarded as the components of a vector. Then the determinant is a *linear function* of that vector. For example,

$$\begin{vmatrix} a & b & c \\ d & e & f \\ sx+tl & sy+tm & sz+tn \end{vmatrix} = s\begin{vmatrix} a & b & c \\ d & e & f \\ x & y & z \end{vmatrix} + t\begin{vmatrix} a & b & c \\ d & e & f \\ l & m & n \end{vmatrix}$$

because the determinant is a linear function of its third row. This and other properties of determinants follow directly from the definition. Some other properties are summarized below. These are stated for rows and for 3×3 determinants, but similar statements can be made for columns and for determinants of any order.

Properties of determinants

(i) If two rows of a determinant are interchanged, then the determinant changes sign:

$$\begin{vmatrix} d & e & f \\ a & b & c \\ g & h & i \end{vmatrix} = -\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}.$$

(ii) If two rows of a determinant are equal, the determinant has value 0:

$$\begin{vmatrix} a & b & c \\ a & b & c \\ g & h & i \end{vmatrix} = 0.$$

(iii) If a multiple of one row of a determinant is added to another row, the value of the determinant remains unchanged:

$$\begin{vmatrix} a & b & c \\ d+ta & e+tb & f+tc \\ g & h & i \end{vmatrix} = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}.$$

The Cross Product as a Determinant

The elements of a determinant are usually numbers because they have to be multiplied to get the value of the determinant. However, it is possible to use vectors as the elements of *one row* (or column) of a determinant. When expanding in minors about that row (or column), the minor for each vector element is a number that determines the scalar multiple of the vector. The formula for the cross product of

$$\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k} \quad \text{and} \quad \mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$$

presented in Theorem 2 can be expressed symbolically as a determinant with the standard basis vectors as the elements of the first row:

$$\mathbf{u}\times\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}\mathbf{k}.$$

The formula for the cross product given in that theorem is just the expansion of this determinant in minors about the first row.

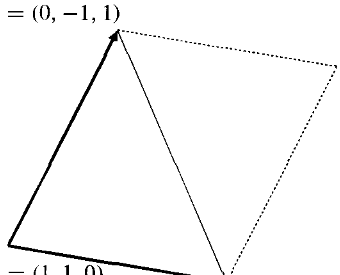


Figure 10.25

Example 3 Find the area of the triangle with vertices at the three points $A = (1, 1, 0)$, $B = (3, 0, 2)$, and $C = (0, -1, 1)$.

Solution Two sides of the triangle (Figure 10.25) are given by the vectors:

$$\overrightarrow{AB} = 2\mathbf{i} - \mathbf{j} + 2\mathbf{k} \quad \text{and} \quad \overrightarrow{AC} = -\mathbf{i} - 2\mathbf{j} + \mathbf{k}.$$

The area of the triangle is half the area of the parallelogram spanned by $\overrightarrow{AB}$ and $\overrightarrow{AC}$. By the definition of cross product, the area of the triangle must therefore be

$$\frac{1}{2}|\overrightarrow{AB}\times\overrightarrow{AC}| = \frac{1}{2}\left|\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 2 \\ -1 & -2 & 1 \end{vmatrix}\right|$$

$$= \frac{1}{2}|3\mathbf{i} - 4\mathbf{j} - 5\mathbf{k}| = \frac{1}{2}\sqrt{9+16+25} = \frac{5}{2}\sqrt{2} \text{ square units.}$$

■

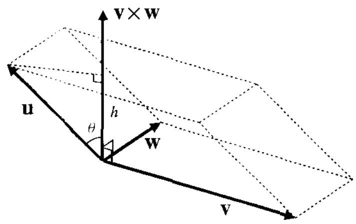


Figure 10.26

A **parallelepiped** is the three-dimensional analogue of a parallelogram. It is a solid with three pairs of parallel planar faces. Each face is in the shape of a parallelogram. A rectangular brick is a special case of a parallelepiped in which nonparallel faces intersect at right angles. We say that a parallelepiped is **spanned** by three vectors coinciding with three of its edges that meet at one vertex. (See Figure 10.26.)

Example 4 Find the volume of the parallelepiped spanned by **u**, **v**, and **w**.

Solution The volume of the parallelepiped is equal to the area of one of its faces, say the face spanned by **v** and **w**, multiplied by the height of the parallelepiped measured in a direction perpendicular to that face. The area of the face is $|\mathbf{v}\times\mathbf{w}|$. Since $\mathbf{v}\times\mathbf{w}$ is perpendicular to the face, the height h of the parallelepiped will be the absolute value of the scalar projection of **u** along $\mathbf{v}\times\mathbf{w}$. If θ is the angle between **u** and $\mathbf{v}\times\mathbf{w}$, then the volume of the parallelepiped is given by

$$\text{Volume} = |\mathbf{u}||\mathbf{v}\times\mathbf{w}||\cos\theta| = |\mathbf{u}\bullet(\mathbf{v}\times\mathbf{w})| \text{ cubic units.}$$

DEFINITION 6

The quantity $\mathbf{u}\bullet(\mathbf{v}\times\mathbf{w})$ is called the **scalar triple product** of the vectors **u**, **v**, and **w**.

The scalar triple product is easily expressed in terms of a determinant. If $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$, and similar representations hold for **v** and **w**, then

$$\begin{aligned}\mathbf{u}\bullet(\mathbf{v}\times\mathbf{w}) &= u_1\begin{vmatrix} v_2 & v_3 \\ w_2 & w_3\end{vmatrix} - u_2\begin{vmatrix} v_1 & v_3 \\ w_1 & w_3\end{vmatrix} + u_3\begin{vmatrix} v_1 & v_2 \\ w_1 & w_2\end{vmatrix} \\ &= \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3\end{vmatrix}.\end{aligned}$$

The volume of the parallelepiped spanned by **u**, **v**, and **w** is the absolute value of this determinant.

Using the properties of the determinant, it is easily verified that

$$\mathbf{u}\bullet(\mathbf{v}\times\mathbf{w}) = \mathbf{v}\bullet(\mathbf{w}\times\mathbf{u}) = \mathbf{w}\bullet(\mathbf{u}\times\mathbf{v}).$$

(See Exercise 18 below.) Note that **u**, **v**, and **w** remain in the same *cyclic order* in these three expressions. Reversing the order would introduce a factor -1:

$$\mathbf{u}\bullet(\mathbf{v}\times\mathbf{w}) = -\mathbf{u}\bullet(\mathbf{w}\times\mathbf{v}).$$

Three vectors in 3-space are said to be **coplanar** if the parallelepiped they span has zero volume; if their tails coincide, three such vectors must lie in the same plane.

$$\begin{aligned}\mathbf{u}, \mathbf{v}, \text{ and } \mathbf{w} \text{ are coplanar} &\iff \mathbf{u}\bullet(\mathbf{v}\times\mathbf{w}) = 0 \\ &\iff \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3\end{vmatrix} = 0.\end{aligned}$$

Three vectors are certainly coplanar if any of them is $\mathbf{0}$, or if any pair of them is parallel. If neither of these degenerate conditions apply, they are only coplanar if any one of them can be expressed as a linear combination of the other two. (See Exercise 20 below.)

Applications of Cross Products

Cross products are of considerable importance in mechanics and electromagnetic theory, as well as in the study of motion in general. For example:

(a) The linear velocity $\mathbf{v}$ of a particle located at position $\mathbf{r}$ in a body rotating with angular velocity $\boldsymbol{\Omega}$ about the origin is given by $\mathbf{v} = \boldsymbol{\Omega}\times\mathbf{r}$. (See Section 11.2 for more details.)

(b) The angular momentum of a planet of mass m moving with velocity $\mathbf{v}$ in its orbit around the sun is given by $\mathbf{h} = \mathbf{r}\times m\mathbf{v}$, where $\mathbf{r}$ is the position vector of the planet relative to the sun as origin. (See Section 11.6.)

(c) If a particle of electric charge q is travelling with velocity $\mathbf{v}$ through a magnetic field whose strength and direction are given by vector $\mathbf{B}$, then the force that the field exerts on the particle is given by $\mathbf{F} = q\mathbf{v}\times\mathbf{B}$. The electron beam in a television tube is controlled by magnetic fields using this principle.

(d) The torque $\mathbf{T}$ of a force $\mathbf{F}$ applied at the point P with position vector $\mathbf{r}$ about another point P_0 with position vector $\mathbf{r}_0$ is defined to be

$$\mathbf{T} = \overrightarrow{P_0P}\times\mathbf{F} = (\mathbf{r} - \mathbf{r}_0)\times\mathbf{F}.$$

This torque measures the effectiveness of the force $\mathbf{F}$ in causing rotation about P_0. The direction of $\mathbf{T}$ is along the axis through P_0 about which $\mathbf{F}$ acts to rotate P.

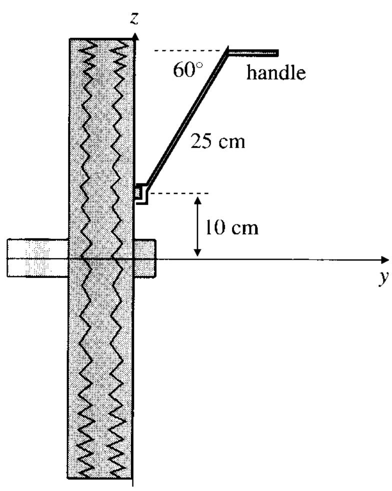


Figure 10.27 The force on the handle is 500 N in a direction directly toward you

Example 5 An automobile wheel has centre at the origin and axle along the y-axis. One of the retaining nuts holding the wheel is at position $P_0 = (0, 0, 10)$. (Distances are measured in centimetres.) A bent tire wrench with arm 25 cm long and inclined at an angle of 60° to the direction of its handle is fitted to the nut in an upright direction, as shown in Figure 10.27. If a horizontal force $\mathbf{F} = 500\mathbf{i}$ newtons (N) is applied to the handle of the wrench, what is its torque on the nut? What part (component) of this torque is effective in trying to rotate the nut about its horizontal axis? What is the effective torque trying to rotate the wheel?

Solution The nut is at position $\mathbf{r}_0 = 10\mathbf{k}$, and the handle of the wrench is at position

$$\mathbf{r} = 25\cos 60°\mathbf{j} + (10 + 25\sin 60°)\mathbf{k} \approx 12.5\mathbf{j} + 31.65\mathbf{k}.$$

The torque of the force $\mathbf{F}$ on the nut is

$$\begin{aligned}\mathbf{T} &= (\mathbf{r} - \mathbf{r}_0)\times\mathbf{F}\\ &\approx (12.5\mathbf{j} + 21.65\mathbf{k})\times 500\mathbf{i} \approx 10{,}825\mathbf{j} - 6{,}250\mathbf{k},\end{aligned}$$

which is at right angles to **F** and to the arm of the wrench. Only the horizontal component of this torque is effective in turning the nut. This component is 10,825 N·cm or 108.25 N·m in magnitude. For the effective torque on the wheel itself, we have to replace $\mathbf{r}_0$ by **0**, the position of the centre of the wheel. In this case the horizontal torque is

$$31.65\mathbf{k}\times 500\mathbf{i} \approx 15{,}825\mathbf{j},$$

that is, about 158.25 N·m.

Exercises 10.3

1. Calculate $\mathbf{u}\times\mathbf{v}$ if $\mathbf{u} = \mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$ and $\mathbf{v} = 3\mathbf{i} + \mathbf{j} - 4\mathbf{k}$.
2. Calculate $\mathbf{u}\times\mathbf{v}$ if $\mathbf{u} = \mathbf{j} + 2\mathbf{k}$ and $\mathbf{v} = -\mathbf{i} - \mathbf{j} + \mathbf{k}$.
3. Find the area of the triangle with vertices $(1, 2, 0)$, $(1, 0, 2)$, and $(0, 3, 1)$.
4. Find a unit vector perpendicular to the plane containing the points $(a, 0, 0)$, $(0, b, 0)$, and $(0, 0, c)$. What is the area of the triangle with these vertices?
5. Find a unit vector perpendicular to the vectors $\mathbf{i} + \mathbf{j}$ and $\mathbf{j} + 2\mathbf{k}$.
6. Find a unit vector with positive $\mathbf{k}$ component that is perpendicular to both $2\mathbf{i} - \mathbf{j} - 2\mathbf{k}$ and $2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$.

Verify the identities in Exercises 7–11, either by using the definition of cross product or the properties of determinants.

7. $\mathbf{u}\times\mathbf{u} = \mathbf{0}$
8. $\mathbf{u}\times\mathbf{v} = -\mathbf{v}\times\mathbf{u}$
9. $(\mathbf{u}+\mathbf{v})\times\mathbf{w} = \mathbf{u}\times\mathbf{w} + \mathbf{v}\times\mathbf{w}$
10. $(t\mathbf{u})\times\mathbf{v} = \mathbf{u}\times(t\mathbf{v}) = t(\mathbf{u}\times\mathbf{v})$
11. $\mathbf{u}\bullet(\mathbf{u}\times\mathbf{v}) = \mathbf{v}\bullet(\mathbf{u}\times\mathbf{v}) = 0$
12. Obtain the addition formula

$$\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$

by examining the cross product of the two unit vectors $\mathbf{u} = \cos\beta\mathbf{i} + \sin\beta\mathbf{j}$ and $\mathbf{v} = \cos\alpha\mathbf{i} + \sin\alpha\mathbf{j}$. Assume $0 \le \alpha - \beta \le \pi$. *Hint:* regard **u** and **v** as position vectors. What is the area of the parallelogram they span?

13. If $\mathbf{u} + \mathbf{v} + \mathbf{w} = \mathbf{0}$, show that $\mathbf{u}\times\mathbf{v} = \mathbf{v}\times\mathbf{w} = \mathbf{w}\times\mathbf{u}$.
14. **(Volume of a tetrahedron)** A **tetrahedron** is a pyramid with a triangular base and three other triangular faces. It has four vertices and six edges. Like any pyramid or cone, its volume is equal to $\frac{1}{3}Ah$, where A is the area of the base and h is the height measured perpendicular to the base. If **u**, **v**, and **w** are vectors coinciding with the three edges of a tetrahedron that meet at one vertex, show that the tetrahedron has volume given by

$$\text{Volume} = \frac{1}{6}|\mathbf{u}\bullet(\mathbf{v}\times\mathbf{w})| = \frac{1}{6}\left|\begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}\right|.$$

Thus, the volume of a tetrahedron spanned by three vectors is one-sixth of the volume of the parallelepiped spanned by the same vectors.

15. Find the volume of the tetrahedron with vertices $(1, 0, 0)$, $(1, 2, 0)$, $(2, 2, 2)$, and $(0, 3, 2)$.
16. Find the volume of the parallelepiped spanned by the diagonals of the three faces of a cube of side a that meet at one vertex of the cube.
17. For what value of k do the four points $(1, 1, -1)$, $(0, 3, -2)$, $(-2, 1, 0)$, and $(k, 0, 2)$ all lie in a plane?
18. **(The scalar triple product)** Verify the identities

$$\mathbf{u}\bullet(\mathbf{v}\times\mathbf{w}) = \mathbf{v}\bullet(\mathbf{w}\times\mathbf{u}) = \mathbf{w}\bullet(\mathbf{u}\times\mathbf{v}).$$

19. If $\mathbf{u}\bullet(\mathbf{v}\times\mathbf{w}) \ne 0$ and **x** is an arbitrary 3-vector, find the numbers λ, μ, and ν such that

$$\mathbf{x} = \lambda\mathbf{u} + \mu\mathbf{v} + \nu\mathbf{w}.$$

20. If $\mathbf{u}\bullet(\mathbf{v}\times\mathbf{w}) = 0$ but $\mathbf{v}\times\mathbf{w} \ne \mathbf{0}$, show that there are constants λ and μ such that

$$\mathbf{u} = \lambda\mathbf{v} + \mu\mathbf{w}.$$

Hint: use the result of Exercise 19 with **u** in place of **x** and $\mathbf{v}\times\mathbf{w}$ in place of **u**.

21. Calculate $\mathbf{u}\times(\mathbf{v}\times\mathbf{w})$ and $(\mathbf{u}\times\mathbf{v})\times\mathbf{w}$, given that $\mathbf{u} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$, $\mathbf{v} = 2\mathbf{i} - 3\mathbf{j}$, and $\mathbf{w} = \mathbf{j} - \mathbf{k}$. Why would you not expect these to be equal?
22. Does the notation $\mathbf{u}\bullet\mathbf{v}\times\mathbf{w}$ make sense? Why? How about the notation $\mathbf{u}\times\mathbf{v}\times\mathbf{w}$?
23. **(The vector triple product)** The product $\mathbf{u}\times(\mathbf{v}\times\mathbf{w})$ is called a **vector triple product**. Since it is perpendicular to $\mathbf{v}\times\mathbf{w}$, it must lie in the plane of **v** and **w**. Show that

$$\mathbf{u}\times(\mathbf{v}\times\mathbf{w}) = (\mathbf{u}\bullet\mathbf{w})\mathbf{v} - (\mathbf{u}\bullet\mathbf{v})\mathbf{w}.$$

Hint: this can be done by direct calculation of the components of both sides of the equation, but the job is much easier if you choose coordinate axes so that $\mathbf{v}$ lies along the x-axis and $\mathbf{w}$ lies in the xy-plane.

24. If $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are mutually perpendicular vectors, show that $\mathbf{u}\times(\mathbf{v}\times\mathbf{w}) = \mathbf{0}$. What is $\mathbf{u} \bullet (\mathbf{v}\times\mathbf{w})$ in this case?

25. Show that $\mathbf{u}\times(\mathbf{v}\times\mathbf{w}) + \mathbf{v}\times(\mathbf{w}\times\mathbf{u}) + \mathbf{w}\times(\mathbf{u}\times\mathbf{v}) = \mathbf{0}$.

26. Find all vectors $\mathbf{x}$ that satisfy the equation

$$(-\mathbf{i} + 2\mathbf{j} + 3\mathbf{k})\times\mathbf{x} = \mathbf{i} + 5\mathbf{j} - 3\mathbf{k}.$$

27. Show that the equation

$$(-\mathbf{i} + 2\mathbf{j} + 3\mathbf{k})\times\mathbf{x} = \mathbf{i} + 5\mathbf{j}$$

has no solutions for the unknown vector $\mathbf{x}$.

28. What condition must be satisfied by the nonzero vectors $\mathbf{a}$ and $\mathbf{b}$ to guarantee that the equation $\mathbf{a}\times\mathbf{x} = \mathbf{b}$ has a solution for $\mathbf{x}$? Is the solution unique?

10.4 Planes and Lines

A single equation in the three variables, x, y, and z, constitutes a single constraint on the freedom of the point $P = (x, y, z)$ to lie anywhere in 3-space. Such a constraint usually results in the loss of exactly one *degree of freedom* and so forces P to lie on a two-dimensional surface. For example, the equation

$$x^2 + y^2 + z^2 = 4$$

states that the point (x, y, z) is at distance 2 from the origin. All points satisfying this condition lie on a **sphere** (i.e., the surface of a ball) of radius 2 centred at the origin. The equation above therefore represents that sphere, and the sphere is the graph of the equation. In this section we will investigate the graphs of linear equations in three variables.

Planes in 3-Space

Let $P_0 = (x_0, y_0, z_0)$ be a point in $\mathbb{R}^3$ with position vector

$$\mathbf{r}_0 = x_0\mathbf{i} + y_0\mathbf{j} + z_0\mathbf{k}.$$

If $\mathbf{n} = A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$ is any given *nonzero* vector, then there exists exactly one **plane** (flat surface) passing through P_0 and perpendicular to $\mathbf{n}$. We say that $\mathbf{n}$ is a **normal vector** to the plane. The plane is the set of all points P for which $\overrightarrow{P_0P}$ is perpendicular to $\mathbf{n}$. (See Figure 10.28.)

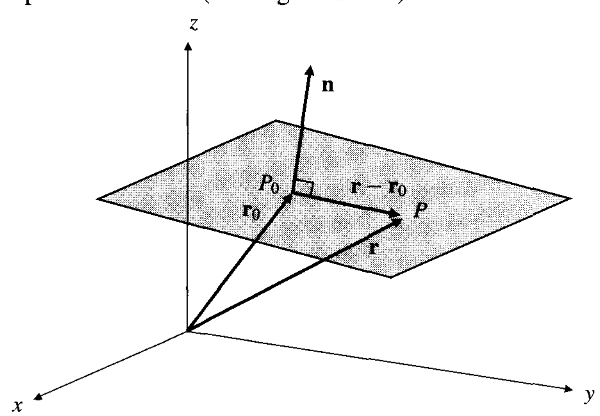


Figure 10.28 The plane through P_0 with normal $\mathbf{n}$ contains all points P for which $\overrightarrow{P_0P}$ is perpendicular to $\mathbf{n}$

If $P = (x, y, z)$ has position vector $\mathbf{r}$, then $\overrightarrow{P_0P} = \mathbf{r} - \mathbf{r}_0$. This vector is perpendicular to $\mathbf{n}$ if and only if $\mathbf{n} \bullet (\mathbf{r} - \mathbf{r}_0) = 0$. This is the equation of the plane in vector form. We can rewrite it in terms of coordinates to obtain the corresponding scalar equation.

The point-normal equation of a plane

The plane having nonzero normal vector $\mathbf{n} = A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$, and passing through the point $P_0 = (x_0, y_0, z_0)$ with position vector $\mathbf{r}_0$, has equation

$$\mathbf{n} \bullet (\mathbf{r} - \mathbf{r}_0) = 0$$

in vector form, or, equivalently,

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$$

in scalar form.

The scalar form can be written more simply in the **standard form** $Ax + By + Cz = D$, where $D = Ax_0 + By_0 + Cz_0$.

If at least one of the constants A, B, and C is not zero, then the *linear equation* $Ax + By + Cz = D$ always represents a plane in $\mathbb{R}^3$. For example, if $A \neq 0$, it represents the plane through $(D/A, 0, 0)$ with normal vector $\mathbf{n} = A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$. A vector normal to a plane can always be determined from the coefficients of x, y, and z. If the constant term $D = 0$, then the plane must pass through the origin.

Example 1 **(Recognizing and writing the equations of planes)**

(a) The equation $2x - 3y - 4z = 0$ represents a plane that passes through the origin and is normal (perpendicular) to the vector $\mathbf{n} = 2\mathbf{i} - 3\mathbf{j} - 4\mathbf{k}$.

(b) The plane that passes through the point $(2, 0, 1)$ and is perpendicular to the straight line passing through the points $(1, 1, 0)$ and $(4, -1, -2)$ has normal vector $\mathbf{n} = (4 - 1)\mathbf{i} + (-1 - 1)\mathbf{j} + (-2 - 0)\mathbf{k} = 3\mathbf{i} - 2\mathbf{j} - 2\mathbf{k}$. Therefore, its equation is $3(x-2)-2(y-0)-2(z-1) = 0$, or, more simply, $3x-2y-2z = 4$.

(c) The plane with equation $2x - y = 1$ has a normal $2\mathbf{i} - \mathbf{j}$ that is perpendicular to the z-axis. The plane is therefore parallel to the z-axis. Note that the equation is independent of z. In the xy-plane, the equation $2x - y = 1$ represents a straight line; in 3-space it represents a plane containing that line and parallel to the z-axis. What does the equation $y = z$ represent in $\mathbb{R}^3$? the equation $y = -2$?

(d) The equation $2x + y + 3z = 6$ represents a plane with normal $\mathbf{n} = 2\mathbf{i} + \mathbf{j} + 3\mathbf{k}$. In this case we cannot directly read from the equation the coordinates of a particular point on the plane, but it is not difficult to discover some points. For instance, if we put $y = z = 0$ in the equation we get $x = 3$, so $(3, 0, 0)$ is a point on the plane. We say that the **x-intercept** of the plane is 3 since $(3, 0, 0)$ is the point where the plane intersects the x-axis. Similarly, the y-intercept is 6 and the z-intercept is 2 because the plane intersects the y- and z-axes at $(0, 6, 0)$ and $(0, 0, 2)$, respectively.

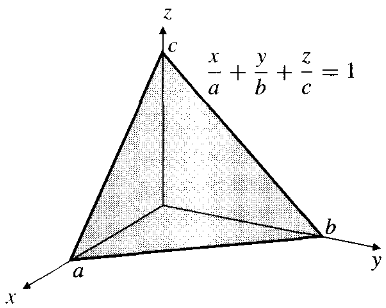


Figure 10.29 The plane with intercepts a, b, and c on the coordinate axes

(e) In general, if a, b, and c are all nonzero, the plane with intercepts a, b, and c on the coordinate axes has equation

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1,$$

called the **intercept form** of the equation of the plane. (See Figure 10.29.) ■

Example 2 Find an equation of the plane that passes through the three points $P=(1,1,0)$, $Q=(0,2,1)$, and $R=(3,2,-1)$.

Solution We need to find a vector, $\mathbf{n}$, normal to the plane. Such a vector will be perpendicular to the vectors $\overrightarrow{PQ}=-\mathbf{i}+\mathbf{j}+\mathbf{k}$ and $\overrightarrow{PR}=2\mathbf{i}+\mathbf{j}-\mathbf{k}$. Therefore, we can use

$$\mathbf{n}=\overrightarrow{PQ}\times\overrightarrow{PR}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ -1 & 1 & 1\\ 2 & 1 & -1\end{vmatrix}=-2\mathbf{i}+\mathbf{j}-3\mathbf{k}.$$

We can use this normal vector together with the coordinates of any one of the three given points to write the equation of the plane. Using point P leads to the equation $-2(x-1)+1(y-1)-3(z-0)=0$, or

$$2x-y+3z=1.$$

You can check that using either Q or R leads to the same equation. (If the cross product $\overrightarrow{PQ}\times\overrightarrow{PR}$ had been the zero vector, what would have been true about the three points P, Q, and R? Would they have determined a unique plane?) ■

Example 3 Show that the two planes

$$x-y=3 \qquad \text{and} \qquad x+y+z=0$$

intersect, and find a vector, $\mathbf{v}$, parallel to their line of intersection.

Solution The two planes have normal vectors

$$\mathbf{n}_1=\mathbf{i}-\mathbf{j} \qquad \text{and} \qquad \mathbf{n}_2=\mathbf{i}+\mathbf{j}+\mathbf{k},$$

respectively. Since these vectors are not parallel, the planes are not parallel, and they intersect in a straight line perpendicular to both $\mathbf{n}_1$ and $\mathbf{n}_2$. This line must therefore be parallel to

$$\mathbf{v}=\mathbf{n}_1\times\mathbf{n}_2=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 1 & -1 & 0\\ 1 & 1 & 1\end{vmatrix}=-\mathbf{i}-\mathbf{j}+2\mathbf{k}.$$

■

A family of planes intersecting in a straight line is called a **pencil of planes**. (See Figure 10.30.) Such a pencil of planes is determined by any two nonparallel planes in it, since these have a unique line of intersection. If the two nonparallel planes have equations

$$A_1x + B_1y + C_1z = D_1 \qquad \text{and} \qquad A_2x + B_2y + C_2z = D_2,$$

then, for any value of the real number λ, the equation

$$A_1x + B_1y + C_1z - D_1 + \lambda(A_2x + B_2y + C_2z - D_2) = 0$$

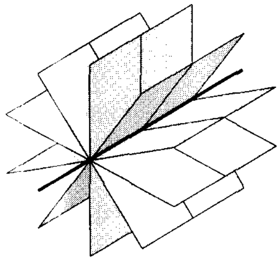


Figure 10.30 A pencil of planes

represents a plane in the pencil. To see this, observe that the equation is linear, and so represents a plane, and that any point (x, y, z) satisfying the equations of both given planes also satisfies this equation for any value of λ. Any plane in the pencil except the second defining plane, $A_2x + B_2y + C_2z = D_2$, can be obtained by suitably choosing λ.

Example 4 Find an equation of the plane passing through the line of intersection of the two planes

$$x + y - 2z = 6 \qquad \text{and} \qquad 2x - y + z = 2$$

and also passing through the point $(-2, 0, 1)$.

Solution For any constant λ, the equation

$$x + y - 2z - 6 + \lambda(2x - y + z - 2) = 0$$

represents a plane and is satisfied by the coordinates of all points on the line of intersection of the given planes. This plane passes through the point $(-2, 0, 1)$ if $-2 - 2 - 6 + \lambda(-4 + 1 - 2) = 0$, that is, if $\lambda = -2$. The equation of the required plane therefore simplifies to $3x - 3y + 4z + 2 = 0$. (This solution would not have worked if the given point had been on the second plane, $2x - y + z = 2$. Why?) ■

Lines in 3-Space

As we observed above, any two nonparallel planes in $\mathbb{R}^3$ determine a unique (straight) line of intersection, and a vector parallel to this line can be obtained by taking the cross product of normal vectors to the two planes.

Suppose that $\mathbf{r}_0 = x_0\mathbf{i} + y_0\mathbf{j} + z_0\mathbf{k}$ is the position vector of point P_0 and $\mathbf{v} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ is a nonzero vector. There is a unique line passing through P_0 parallel to $\mathbf{v}$. If $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ is the position vector of any other point P on the line, then $\mathbf{r} - \mathbf{r}_0$ lies along the line and so is parallel to $\mathbf{v}$. (See Figure 10.31.) Thus, $\mathbf{r} - \mathbf{r}_0 = t\mathbf{v}$ for some real number t. This equation, usually rewritten in the form

$$\mathbf{r} = \mathbf{r}_0 + t\mathbf{v},$$

is called the **vector parametric equation of the straight line**. All points on the line can be obtained as the parameter t ranges from $-\infty$ to ∞. The vector $\mathbf{v}$ is called a **direction vector** of the line.

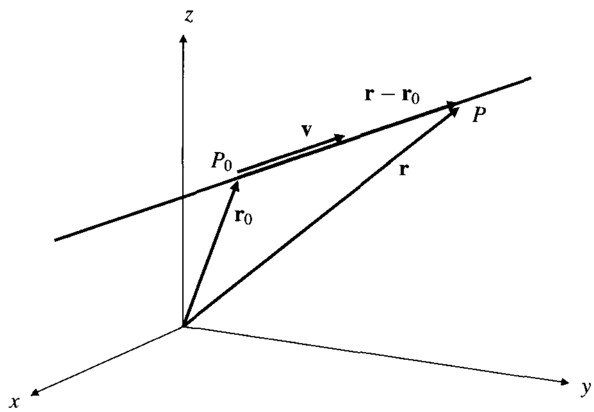


Figure 10.31 The line through P_0 parallel to $\mathbf{v}$

Breaking the vector parametric equation down into its components yields the **scalar parametric** equations of the line:

$$\begin{cases} x = x_0 + at \\ y = y_0 + bt \\ z = z_0 + ct. \end{cases} \qquad (-\infty < t < \infty)$$

These appear to be *three* linear equations, but the parameter t can be eliminated to give *two* linear equations in x, y, and z. If $a \neq 0$, $b \neq 0$, and $c \neq 0$, then we can solve each of the scalar equations for t and so obtain

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c},$$

which is called the **standard form** for the equations of the straight line through (x_0, y_0, z_0) parallel to $\mathbf{v}$. The standard form must be modified if any component of $\mathbf{v}$ vanishes. For example, if $c = 0$, the equations are

$$\frac{x - x_0}{a} = \frac{y - y_0}{b}, \qquad z = z_0.$$

Note that none of the above equations for straight lines is unique; each depends on the particular choice of the point (x_0, y_0, z_0) on the line. In general, you can always use the equations of two nonparallel planes to represent their line of intersection.

Example 5 **(Equations of straight lines)**

(a) The equations

$$\begin{cases} x = 2 + t \\ y = 3 \\ z = -4t \end{cases}$$

represent the straight line through $(2, 3, 0)$ parallel to the vector $\mathbf{i} - 4\mathbf{k}$.

(b) The straight line through $(1, -2, 3)$ perpendicular to the plane $x - 2y + 4z = 5$ is parallel to the normal vector $\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}$ of the plane. Therefore, the line has vector parametric equation

$$\mathbf{r} = \mathbf{i} - 2\mathbf{j} + 3\mathbf{k} + t(\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}),$$

or scalar parametric equations

$$\begin{cases} x = 1 + t \\ y = -2 - 2t \\ z = 3 + 4t. \end{cases}$$

Its standard form equations are

$$\frac{x-1}{1} = \frac{y+2}{-2} = \frac{z-3}{4}.$$

Example 6 Find a direction vector for the line of intersection of the two planes

$$x + y - z = 0 \qquad \text{and} \qquad y + 2z = 6,$$

and find a set of equations for the line in standard form.

Solution The two planes have respective normals $\mathbf{n}_1 = \mathbf{i}+\mathbf{j}-\mathbf{k}$ and $\mathbf{n}_2 = \mathbf{j}+2\mathbf{k}$. Thus, a direction vector of their line of intersection is

$$\mathbf{v} = \mathbf{n}_1 \times \mathbf{n}_2 = 3\mathbf{i} - 2\mathbf{j} + \mathbf{k}.$$

We need to know one point on the line in order to write equations in standard form. We can find a point by assigning a value to one coordinate and calculating the other two from the given equations. For instance, taking $z = 0$ in the two equations we are led to $y = 6$ and $x = -6$, so $(-6, 6, 0)$ is one point on the line. Thus, the line has standard form equations

$$\frac{x+6}{3} = \frac{y-6}{-2} = z.$$

This answer is not unique; the coordinates of any other point on the line could be used in place of $(-6, 6, 0)$. You could even find a direction vector $\mathbf{v}$ by subtracting the position vectors of two different points on the line.

Distances

The **distance** between two geometric objects always means the minimum distance between two points, one in each object. In the case of *flat* objects like lines or planes defined by linear equations, such minimum distances can usually be determined by geometric arguments without having to use calculus.

Example 7 **(Distance from a point to a plane)**

(a) Find the distance from the point $P_0 = (x_0, y_0, z_0)$ to the plane $\mathcal{P}$ having equation $Ax + By + Cz = D$.

(b) What is the distance from $(2, -1, 3)$ to the plane $2x - 2y - z = 9$?

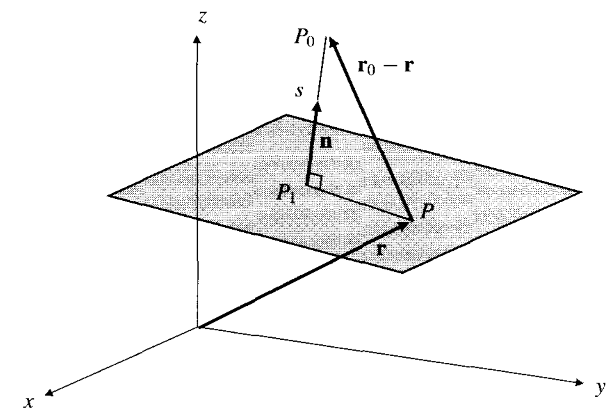


Figure 10.32 The distance from P_0 to the plane $\mathcal{P}$ is the length of the vector projection of PP_0 along the normal $\mathbf{n}$ to $\mathcal{P}$, where P is any point on $\mathcal{P}$

Solution

(a) Let $\mathbf{r}_0$ be the position vector of P_0 and let $\mathbf{n} = A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$ be the normal to $\mathcal{P}$. Let P_1 be the point on $\mathcal{P}$ that is closest to P_0. Then $\overrightarrow{P_1P_0}$ is perpendicular to $\mathcal{P}$ and so is parallel to $\mathbf{n}$. The distance from P_0 to $\mathcal{P}$ is $s = |\overrightarrow{P_1P_0}|$. If P, having position vector $\mathbf{r}$, is any point on $\mathcal{P}$, then s is the length of the projection of $\overrightarrow{PP_0} = \mathbf{r}_0 - \mathbf{r}$ in the direction of $\mathbf{n}$. (See Figure 10.32.) Thus

$$s = \left| \frac{\overrightarrow{PP_0} \bullet \mathbf{n}}{|\mathbf{n}|} \right| = \frac{|(\mathbf{r}_0 - \mathbf{r}) \bullet \mathbf{n}|}{|\mathbf{n}|} = \frac{|\mathbf{r}_0 \bullet \mathbf{n} - \mathbf{r} \bullet \mathbf{n}|}{|\mathbf{n}|}.$$

Since $P = (x, y, z)$ lies on $\mathcal{P}$, we have $\mathbf{r} \bullet \mathbf{n} = Ax + By + Cz = D$. In terms of the coordinates (x_0, y_0, z_0) of P_0, we can therefore represent the distance from P_0 to $\mathcal{P}$ as

$$s = \frac{|Ax_0 + By_0 + Cz_0 - D|}{\sqrt{A^2 + B^2 + C^2}}.$$

(b) The distance from $(2, -1, 3)$ to the plane $2x - 2y - z = 9$ is

$$s = \frac{|2(2) - 2(-1) - 1(3) - 9|}{\sqrt{2^2 + (-2)^2 + (-1)^2}} = \frac{|-6|}{3} = 2 \text{ units.}$$

■

Example 8 **(Distance from a point to a line)**

(a) Find the distance from the point P_0 to the straight line $\mathcal{L}$ through P_1 parallel to the nonzero vector $\mathbf{v}$.

(b) What is the distance from $(2, 0, -3)$ to the line $\mathbf{r} = \mathbf{i} + (1 + 3t)\mathbf{j} - (3 - 4t)\mathbf{k}$?

Solution

(a) Let $\mathbf{r}_0$ and $\mathbf{r}_1$ be the position vectors of P_0 and P_1, respectively. The point P_2 on $\mathcal{L}$ that is closest to P_0 is such that P_2P_0 is perpendicular to $\mathcal{L}$. The distance from P_0 to $\mathcal{L}$ is

$$s = |P_2P_0| = |P_1P_0|\sin\theta = |\mathbf{r}_0 - \mathbf{r}_1|\sin\theta,$$

where θ is the angle between $\mathbf{r}_0 - \mathbf{r}_1$ and $\mathbf{v}$. (See Figure 10.33(a).) Since

$$|(\mathbf{r}_0 - \mathbf{r}_1)\times\mathbf{v}| = |\mathbf{r}_0 - \mathbf{r}_1|\,|\mathbf{v}|\,\sin\theta,$$

we have

$$s = \frac{|(\mathbf{r}_0 - \mathbf{r}_1)\times\mathbf{v}|}{|\mathbf{v}|}.$$

(b) The line $\mathbf{r} = \mathbf{i} + (1+3t)\mathbf{j} - (3-4t)\mathbf{k}$ passes through $P_1 = (1, 1, -3)$ and is parallel to $\mathbf{v} = 3\mathbf{j} + 4\mathbf{k}$. The distance from $P_0 = (2, 0, -3)$ to this line is

$$s = \frac{\big|\big((2-1)\mathbf{i} + (0-1)\mathbf{j} + (-3+3)\mathbf{k}\big)\times(3\mathbf{j}+4\mathbf{k})\big|}{\sqrt{3^2+4^2}}$$
$$= \frac{|(\mathbf{i}-\mathbf{j})\times(3\mathbf{j}+4\mathbf{k})|}{5} = \frac{|-4\mathbf{i}-4\mathbf{j}+3\mathbf{k}|}{5} = \frac{\sqrt{41}}{5}\text{ units.}$$

■

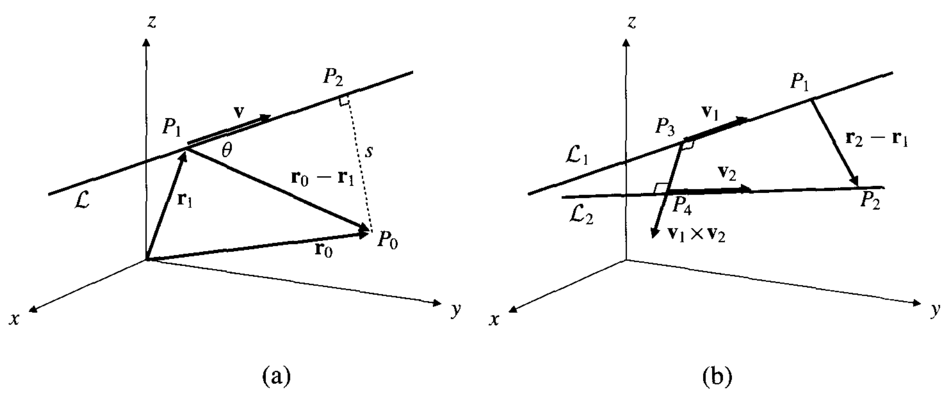


Figure 10.33

(a) The distance from P_0 to the line $\mathcal{L}$ is $s = |P_0P_1|\sin\theta$

(b) The distance between the lines $\mathcal{L}_1$ and $\mathcal{L}_2$ is the length of the projection of P_1P_2 along the vector $\mathbf{v}_1\times\mathbf{v}_2$

Example 9 **(The distance between two lines)** Find the distance between the two lines $\mathcal{L}_1$ through point P_1 parallel to vector $\mathbf{v}_1$ and $\mathcal{L}_2$ through point P_2 parallel to vector $\mathbf{v}_2$.

Solution Let $\mathbf{r}_1$ and $\mathbf{r}_2$ be the position vectors of points P_1 and P_2, respectively. If P_3 and P_4 (with position vectors $\mathbf{r}_3$ and $\mathbf{r}_4$) are the points on $\mathcal{L}_1$ and $\mathcal{L}_2$, respectively, that are closest to one another, then $\overrightarrow{P_3P_4}$ is perpendicular to both lines and is therefore parallel to $\mathbf{v}_1\times\mathbf{v}_2$. (See Figure 10.33(b).) $\overrightarrow{P_3P_4}$ is the vector projection of $\overrightarrow{P_1P_2} = \mathbf{r}_2 - \mathbf{r}_1$ along $\mathbf{v}_1\times\mathbf{v}_2$. Therefore, the distance $s = |\overrightarrow{P_3P_4}|$ between the lines is given by

$$s = |\mathbf{r}_4 - \mathbf{r}_3| = \frac{|(\mathbf{r}_2 - \mathbf{r}_1)\bullet(\mathbf{v}_1\times\mathbf{v}_2)|}{|\mathbf{v}_1\times\mathbf{v}_2|}.$$

■

Exercises 10.4

1. A single equation involving the coordinates (x, y, z) need not always represent a two-dimensional "surface" in $\mathbb{R}^3$. For example, $x^2 + y^2 + z^2 = 0$ represents the single point $(0, 0, 0)$, which has dimension zero. Give examples of single equations in x, y, and z that represent
   (a) a (one-dimensional) straight line,
   (b) the whole of $\mathbb{R}^3$,
   (c) no points at all (i.e., the empty set).

In Exercises 2–9, find equations of the planes satisfying the given conditions.

2. Passing through $(0, 2, -3)$ and normal to the vector $4\mathbf{i} - \mathbf{j} - 2\mathbf{k}$
3. Passing through the origin and having normal $\mathbf{i} - \mathbf{j} + 2\mathbf{k}$
4. Passing through $(1, 2, 3)$ and parallel to the plane $3x + y - 2z = 15$
5. Passing through the three points $(1, 1, 0)$, $(2, 0, 2)$, and $(0, 3, 3)$
6. Passing through the three points $(-2, 0, 0)$, $(0, 3, 0)$, and $(0, 0, 4)$
7. Passing through $(1, 1, 1)$ and $(2, 0, 3)$ and perpendicular to the plane $x + 2y - 3z = 0$
8. Passing through the line of intersection of the planes $2x + 3y - z = 0$ and $x - 4y + 2z = -5$, and passing through the point $(-2, 0, -1)$
9. Passing through the line $x + y = 2$, $y - z = 3$, and perpendicular to the plane $2x + 3y + 4z = 5$
10. Under what geometric condition will three distinct points in $\mathbb{R}^3$ not determine a unique plane passing through them? How can this condition be expressed algebraically in terms of the position vectors, $\mathbf{r}_1$, $\mathbf{r}_2$, and $\mathbf{r}_3$, of the three points?
11. Give a condition on the position vectors of four points that guarantees that the four points are *coplanar*, that is, all lie on one plane.

Describe geometrically the one-parameter families of planes in Exercises 12–14. (λ is a real parameter.)

12. $x + y + z = \lambda$.
*13. $x + \lambda y + \lambda z = \lambda$.
*14. $\lambda x + \sqrt{1 - \lambda^2}\, y = 1$.

In Exercises 15–19, find equations of the line specified in vector and scalar parametric forms and in standard form.

15. Through the point $(1, 2, 3)$ and parallel to $2\mathbf{i} - 3\mathbf{j} - 4\mathbf{k}$
16. Through $(-1, 0, 1)$ and perpendicular to the plane $2x - y + 7z = 12$
17. Through the origin and parallel to the line of intersection of the planes $x + 2y - z = 2$ and $2x - y + 4z = 5$
18. Through $(2, -1, -1)$ and parallel to each of the two planes $x + y = 0$ and $x - y + 2z = 0$
19. Through $(1, 2, -1)$ and making equal angles with the positive directions of the coordinate axes

In Exercises 20–22, find the equations of the given line in standard form.

20. $\mathbf{r} = (1 - 2t)\mathbf{i} + (4 + 3t)\mathbf{j} + (9 - 4t)\mathbf{k}$.

21. $\begin{cases} x = 4 - 5t \\ y = 3t \\ z = 7 \end{cases}$

22. $\begin{cases} x - 2y + 3z = 0 \\ 2x + 3y - 4z = 4 \end{cases}$

23. If $P_1 = (x_1, y_1, z_1)$ and $P_2 = (x_2, y_2, z_2)$, show that the equations

$$\begin{cases} x = x_1 + t(x_2 - x_1) \\ y = y_1 + t(y_2 - y_1) \\ z = z_1 + t(z_2 - z_1) \end{cases}$$

represent a line through P_1 and P_2.

24. What points on the line in the previous exercise correspond to the parameter values $t = -1$, $t = 1/2$, and $t = 2$? Describe their locations.
25. Under what conditions on the position vectors of four distinct points P_1, P_2, P_3, and P_4 will the straight line through P_1 and P_2 intersect the straight line through P_3 and P_4 at a unique point?

Find the required distances in Exercises 26–29.

26. From the origin to the plane $x + 2y + 3z = 4$
27. From $(1, 2, 0)$ to the plane $3x - 4y - 5z = 2$
28. From the origin to the line $x + y + z = 0$, $2x - y - 5z = 1$
29. Between the lines

$$\begin{cases} x + 2y = 3 \\ y + 2z = 3 \end{cases} \quad \text{and} \quad \begin{cases} x + y + z = 6 \\ x - 2z = -5 \end{cases}$$

30. Show that the line $x - 2 = \dfrac{y + 3}{2} = \dfrac{z - 1}{4}$ is parallel to the plane $2y - z = 1$. What is the distance between the line and the plane?

In Exercises 31–32, describe the one-parameter families of straight lines represented by the given equations. (λ is a real parameter.)

*31. $(1 - \lambda)(x - x_0) = \lambda(y - y_0)$, $z = z_0$.

*32. $\dfrac{x - x_0}{\sqrt{1 - \lambda^2}} = \dfrac{y - y_0}{\lambda} = z - z_0$.

33. Why does the factored second-degree equation

$$(A_1x + B_1y + C_1z - D_1)(A_2x + B_2y + C_2z - D_2) = 0$$

represent a pair of planes rather than a single straight line?

10.5 Quadric Surfaces

The most general second-degree equation in three variables is

$$Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + Gx + Hy + Iz = J.$$

We will not attempt the (rather difficult) task of classifying all the surfaces that can be represented by such an equation, but will examine some interesting special cases. Let us observe at the outset that if the above equation can be factored in the form

$$(A_1x + B_1y + C_1z - D_1)(A_2x + B_2y + C_2z - D_2) = 0,$$

quadric surface) will not be flat, although there may still be straight lines that lie on the surface. Nondegenerate quadric surfaces fall into the following six categories.

Spheres. The equation $x^2 + y^2 + z^2 = a^2$ represents a sphere of radius a centred at the origin. More generally,

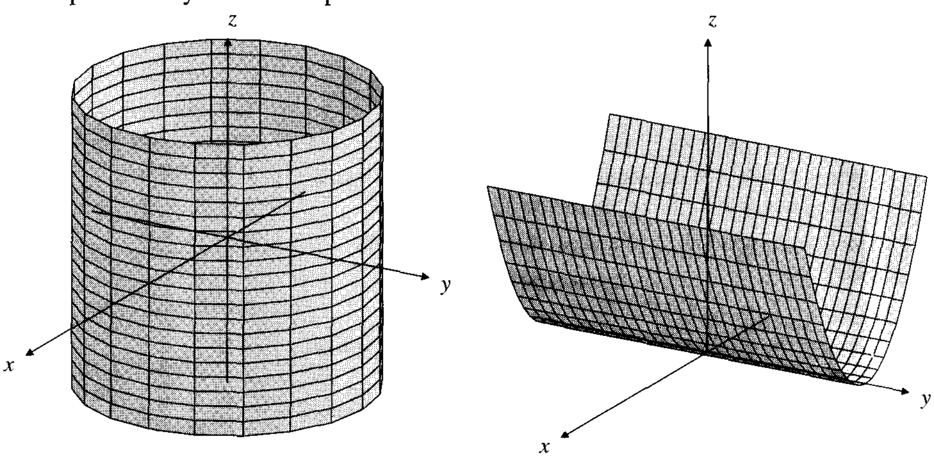
$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = a^2$$

represents a sphere of radius a centred at the point (x_0, y_0, z_0). If a quadratic equation in x, y, and z has equal coefficients for the x^2, y^2, and z^2 terms and has no other second-degree terms, then it will represent, if any surface at all, a sphere. The centre can be found by completing the squares as for circles in the plane.

Cylinders. The equation $x^2 + y^2 = a^2$, being independent of z, represents a **right-circular cylinder** of radius a and axis along the z-axis. (See Figure 10.34(a).) The intersection of the cylinder with the horizontal plane $z = k$ is the circle with equations

$$\begin{cases} x^2 + y^2 = a^2 \\ z = k. \end{cases}$$

Quadric cylinders also come in other shapes: elliptic, parabolic, and hyperbolic. For instance, $z = x^2$ represents a parabolic cylinder with vertex line along the y-axis. (See Figure 10.34(b).) In general, an equation in two variables only will represent a cylinder in 3-space.



(a) (b)

Figure 10.34

(a) The circular cylinder $x^2 + y^2 = a^2$

(b) The parabolic cylinder $z = x^2$

Cones. The equation $z^2 = x^2 + y^2$ represents a **right-circular cone** with axis along the z-axis. The surface is generated by rotating about the z-axis the line $z = y$ in the yz-plane. This *generator* makes an angle of 45° with the axis of the cone. Cross-sections of the cone in planes parallel to the xy-plane are circles. (See Figure 10.35(a).) The equation $x^2 + y^2 = a^2z^2$ also represents a right-circular cone with vertex at the origin and axis along the z-axis but having semi-vertical angle $\alpha = \tan^{-1} a$. A circular cone has plane cross-sections that are elliptical, parabolic, and hyperbolic. Conversely, any nondegenerate quadric cone has a direction perpendicular to which the cross-sections of the cone are circular. In that sense, every quadric cone is a circular cone, though it may be *oblique* rather than right-circular in that the line joining the centres of the circular cross-sections need not be perpendicular to those cross-sections. (See Exercise 24.)

Ellipsoids. The equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

represents an **ellipsoid** with *semi-axes* a, b, and c. (See Figure 10.35(b).) The surface is oval, and it is enclosed inside the rectangular parallelepiped $-a \le x \le a$, $-b \le y \le b$, $-c \le z \le c$. If $a = b = c$, the ellipsoid is a sphere. In general, all plane cross-sections of ellipsoids are ellipses. This is easy to see for cross-sections parallel to coordinate planes, but somewhat harder to see for other planes.

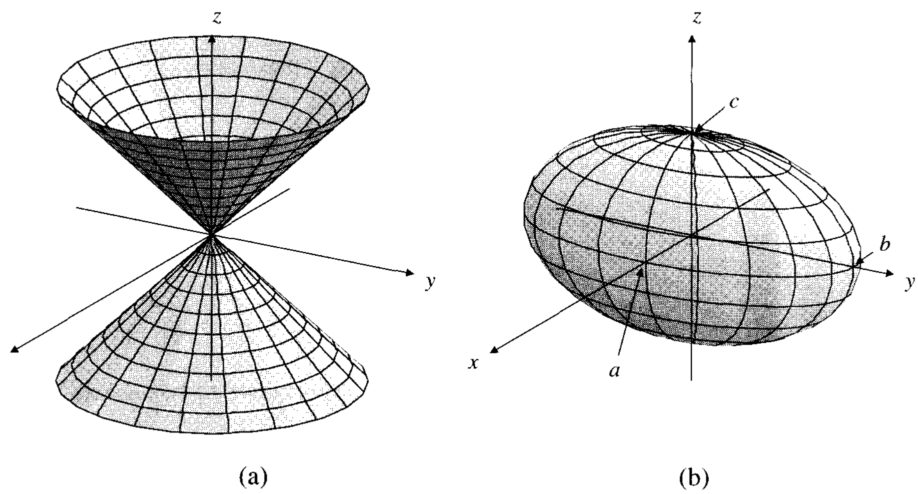


Figure 10.35
(a) The circular cone $a^2z^2 = x^2 + y^2$
(b) The ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

Paraboloids. The equations

$$z = \frac{x^2}{a^2} + \frac{y^2}{b^2} \qquad \text{and} \qquad z = \frac{x^2}{a^2} - \frac{y^2}{b^2}$$

represent, respectively, an **elliptic paraboloid** and a **hyperbolic paraboloid**. (See Figure 10.36(a) and (b).) Cross-sections in planes $z = k$ (k being a positive constant) are ellipses (circles if $a = b$) and hyperbolas, respectively. Parabolic reflective mirrors have the shape of circular paraboloids. The hyperbolic paraboloid is a **ruled surface**. (A ruled surface is one through every point of which there passes a straight line lying wholly on the surface. Cones and cylinders are also examples of ruled surfaces.) There are two one-parameter families of straight lines that lie on the hyperbolic paraboloid, namely,

$$\begin{cases} \lambda z = \dfrac{x}{a} - \dfrac{y}{b} \\ \dfrac{1}{\lambda} = \dfrac{x}{a} + \dfrac{y}{b} \end{cases} \quad \text{and} \quad \begin{cases} \mu z = \dfrac{x}{a} + \dfrac{y}{b} \\ \dfrac{1}{\mu} = \dfrac{x}{a} - \dfrac{y}{b} \end{cases}$$

where λ and μ are real parameters. Every point on the hyperbolic paraboloid lies on one line of each family.

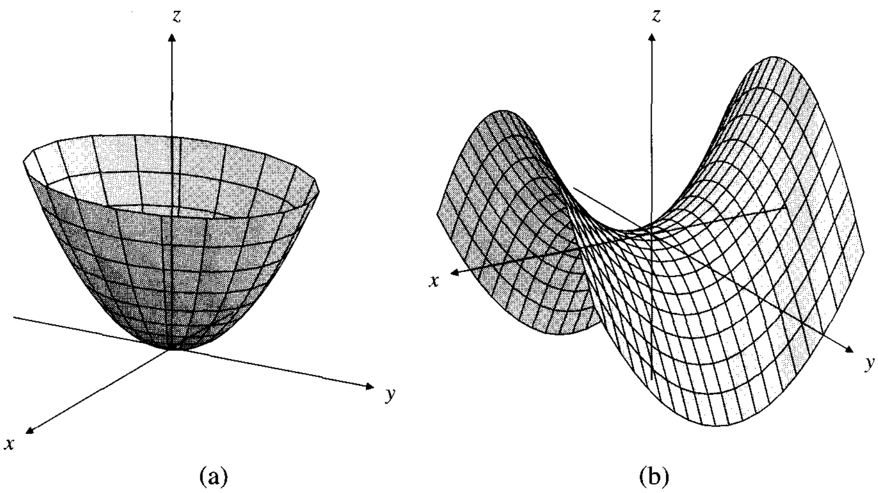


Figure 10.36
(a) The elliptic paraboloid $z = \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2}$
(b) The hyperbolic paraboloid $z = \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2}$

Hyperboloids. The equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

represents a surface called a **hyperboloid of one sheet**. (See Figure 10.37(a).) The equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = -1$$

represents a **hyperboloid of two sheets**. (See Figure 10.37(b).) Both surfaces

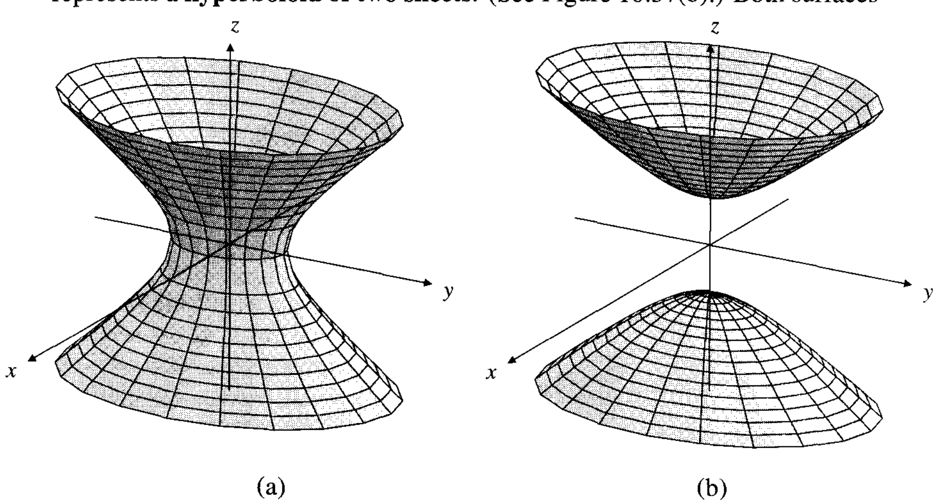


Figure 10.37
(a) The hyperboloid of one sheet $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = 1$
(b) The hyperboloid of two sheets $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = -1$

have elliptical cross-sections in horizontal planes and hyperbolic cross-sections in vertical planes. Both are *asymptotic* to the elliptic cone with equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2};$$

they approach arbitrarily close to the cone as they recede arbitrarily far away from the origin. Like the hyperbolic paraboloid, the hyperboloid of one sheet is a ruled surface.

Exercises 10.5

Identify the surfaces represented by the equations in Exercises 1–16 and sketch their graphs.

1. $x^2 + 4y^2 + 9z^2 = 36$

2. $x^2 + y^2 + 4z^2 = 4$

3. $2x^2 + 2y^2 + 2z^2 - 4x + 8y - 12z + 27 = 0$

4. $x^2 + 4y^2 + 9z^2 + 4x - 8y = 8$

5. $z = x^2 + 2y^2$

6. $z = x^2 - 2y^2$

7. $x^2 - y^2 - z^2 = 4$

8. $-x^2 + y^2 + z^2 = 4$

9. $z = xy$

10. $x^2 + 4z^2 = 4$

11. $x^2 - 4z^2 = 4$

12. $y = z^2$

13. $x = z^2 + z$

14. $x^2 = y^2 + 2z^2$

15. $(z-1)^2 = (x-2)^2 + (y-3)^2$

16. $(z-1)^2 = (x-2)^2 + (y-3)^2 + 4$

Describe and sketch the geometric objects represented by the systems of equations in Exercises 17–20.

17. $\begin{cases} x^2 + y^2 + z^2 = 4 \\ x + y + z = 1 \end{cases}$

18. $\begin{cases} x^2 + y^2 = 1 \\ z = x + y \end{cases}$

19. $\begin{cases} z^2 = x^2 + y^2 \\ z = 1 + x \end{cases}$

20. $\begin{cases} x^2 + 2y^2 + 3z^2 = 6 \\ y = 1 \end{cases}$

21. Find two one-parameter families of straight lines that lie on the hyperboloid of one sheet

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1.$$

22. Find two one-parameter families of straight lines that lie on the hyperbolic paraboloid $z = xy$.

23. The equation $2x^2 + y^2 = 1$ represents a cylinder with elliptical cross-sections in planes perpendicular to the z-axis. Find a vector **a** perpendicular to which the cylinder has circular cross-sections.

***24.** The equation $z^2 = 2x^2 + y^2$ represents a cone with elliptical cross-sections in planes perpendicular to the z-axis. Find a vector **a** perpendicular to which the cone has circular cross-sections. *Hint:* do Exercise 23 first and use its result.

10.6 A Little Linear Algebra

Differential calculus is essentially the study of linear approximations to functions. The tangent line to the graph $y = f(x)$ at $x = x_0$ provides the "best linear approximation" to $f(x)$ near x_0. Differentiation of functions of several variables can also be viewed as a process of finding *best linear approximations.* Therefore the language of linear algebra can be very useful for expressing certain concepts in the calculus of several variables.

Linear algebra is a vast subject and is usually studied independently of calculus. This is unfortunate because understanding the relationship between the two subjects can greatly enhance your understanding and appreciation of each of them. Knowledge of linear algebra, and therefore familiarity with the material covered in this section, is *not essential* for fruitful study of the rest of this book. However, we shall from time to time comment on the significance of the subject at hand from the point of view of linear algebra. To this end we need only a little of the terminology and content of linear algebra, especially that part pertaining to matrix manipulation and systems of linear equations. In the rest of this section we present an outline of this material. Some students will already be familiar with it; others will encounter it later. We make no attempt at completeness here and refer interested students to standard linear algebra texts for proofs of some assertions. Students proceeding beyond this book to further study of advanced calculus and differential equations will certainly need a much more extensive background in linear algebra.

Matrices

An $m \times n$ **matrix** $\mathcal{A}$ is a rectangular array of mn numbers arranged in m rows and n columns. If a_{ij} is the element in the ith row and the jth column, then

$$\mathcal{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}.$$

Sometimes, as a shorthand notation, we write $\mathcal{A} = (a_{ij})$. In this case i is assumed to range from 1 to m and j from 1 to n. If $m = n$, we say that $\mathcal{A}$ is a square matrix. The elements a_{ij} of the matrices we use in this book will always be real numbers.

The **transpose** of an $m \times n$ matrix $\mathcal{A}$ is the $n \times m$ matrix $\mathcal{A}^T$ whose rows are the columns of $\mathcal{A}$:

$$\mathcal{A}^T = \begin{pmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{pmatrix}.$$

Matrix $\mathcal{A}$ is called **symmetric** if $\mathcal{A}^T = \mathcal{A}$. Symmetric matrices are necessarily square. Observe that $(\mathcal{A}^T)^T = \mathcal{A}$ for *every* matrix $\mathcal{A}$. Frequently we want to consider an n-vector $\mathbf{x}$ as an $n \times 1$ matrix having n rows and one column:

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}.$$

As such, $\mathbf{x}$ is called a **column vector**. $\mathbf{x}^T$ then has one row and n columns and is called a **row vector**:

$$\mathbf{x}^T = (x_1\ x_2\ \cdots\ x_n).$$

Note that $\mathbf{x}$ and $\mathbf{x}^T$ have the same components, so they are identical as vectors even though they appear differently as matrices.

Most of the usefulness of matrices depends on the following definition of matrix multiplication, which enables two arrays to be combined into a single one in a manner that preserves linear relationships.

DEFINITION 7

Multiplying matrices

If $\mathcal{A} = (a_{ij})$ is an $m \times n$ matrix and $\mathcal{B} = (b_{ij})$ is an $n \times p$ matrix, then the product $\mathcal{AB}$ is the $m \times p$ matrix $\mathcal{C} = (c_{ij})$ with elements given by

$$c_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj}, \qquad i = 1, \dots, m, \quad j = 1, \dots, p.$$

That is, c_{ij} is the *dot product* of the ith row of $\mathcal{A}$ and the jth column of $\mathcal{B}$ (both of which are n-vectors).

Note that only *some* pairs of matrices can be multiplied. The product $\mathcal{AB}$ is only defined if the number of columns of $\mathcal{A}$ is equal to the number of rows of $\mathcal{B}$.

Example 1

$$\begin{pmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \end{pmatrix} \begin{pmatrix} 2 & 1 & 1 & 0 \\ 0 & -1 & 3 & 1 \\ 1 & 0 & 4 & 5 \end{pmatrix} = \begin{pmatrix} 5 & 1 & 13 & 15 \\ 3 & 1 & 1 & -4 \end{pmatrix}$$

The left factor has 2 rows and 3 columns, and the right factor has 3 rows and 4 columns. Therefore the product has 2 rows and 4 columns. The element in the first row and third column of the product, 13, is the dot product of the first row, $(1, 0, 3)$, of the left factor and the third column, $(1, 3, 4)$, of the second factor:

$$1 \times 1 + 0 \times 3 + 3 \times 4 = 13.$$

With a little practice you can easily calculate the elements of a matrix product by simultaneously running your left index finger across rows of the left factor and your right index finger down columns of the right factor while taking the dot products. ■

Example 2

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & -1 \\ -2 & 3 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x + 2y + 3z \\ y - z \\ -2x + 3y \end{pmatrix}$$

The product of a 3×3 matrix with a column 3-vector is a column 3-vector. ■

Matrix multiplication is *associative.* This means that

$$\mathcal{A}(\mathcal{BC}) = (\mathcal{AB})\mathcal{C}$$

(provided $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ have dimensions compatible with the formation of the various products); therefore it makes sense to write $\mathcal{ABC}$. However, matrix multiplication is *not commutative.* Indeed, if $\mathcal{A}$ is an $m \times n$ matrix and $\mathcal{B}$ is an $n \times p$ matrix, then the product $\mathcal{AB}$ is defined, but the product $\mathcal{BA}$ is not defined unless $m = p$. Even if $\mathcal{A}$ and $\mathcal{B}$ are square matrices of the same size, it is not necessarily true that $\mathcal{AB} = \mathcal{BA}$.

Example 3

$$\begin{pmatrix} 1 & 2 \\ 3 & 0 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 3 & -3 \end{pmatrix} \quad \text{but} \quad \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 0 \end{pmatrix} = \begin{pmatrix} -2 & 2 \\ 4 & 2 \end{pmatrix}$$

Determinants and Matrix Inverses

In Section 10.3 we introduced 2×2 and 3×3 determinants as certain algebraic expressions associated with 2×2 and 3×3 square arrays of numbers. In general, it is possible to define the determinant $\det(\mathcal{A})$ for any square matrix. For an $n \times n$ matrix $\mathcal{A}$ we continue to denote

$$\det(\mathcal{A}) = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}.$$

We will not attempt to give a formal definition of the determinant here but will note that the properties of determinants stated for the 3×3 case in Section 10.3 continue to be true. In particular, an $n \times n$ determinant can be expanded in minors about any row or column and so expressed as a sum of multiples of $(n-1) \times (n-1)$ determinants. Continuing this process, we can eventually reduce the evaluation of any $n \times n$ determinant to the evaluation of (perhaps many) 2×2 or 3×3 determinants. It is important to realize that the "diagonal" method for evaluating 2×2 or 3×3 determinants does not extend to 4×4 or higher-order determinants.

Example 4

$$\begin{aligned}\begin{vmatrix} 2 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 3 & 0 & 0 & 2 \\ -1 & 1 & 1 & 0 \end{vmatrix} &= -\begin{vmatrix} 2 & 1 & 1 \\ 3 & 0 & 2 \\ -1 & 1 & 0 \end{vmatrix} - \begin{vmatrix} 2 & 1 & 1 \\ 1 & 0 & 1 \\ 3 & 0 & 2 \end{vmatrix} \\ &= -\left(-3\begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} - 2\begin{vmatrix} 2 & 1 \\ -1 & 1 \end{vmatrix}\right) - \left(-1\begin{vmatrix} 1 & 1 \\ 3 & 2 \end{vmatrix}\right) \\ &= 3(0-1) + 2(2+1) + 1(2-3) = 2.\end{aligned}$$

We expanded the 4×4 determinant in minors about the third column to obtain the two 3×3 determinants. The first of these was expanded about the second row, the other about the second column.

In addition to the properties stated in Section 10.3, determinants have two other very important properties, which are stated in the following theorem.

THEOREM 3

If $\mathcal{A}$ and $\mathcal{B}$ are $n \times n$ matrices, then

(a) $\det(\mathcal{A}^T) = \det(\mathcal{A})$ and

(b) $\det(\mathcal{AB}) = \det(\mathcal{A})\det(\mathcal{B})$.

We will not attempt any proof of this or other theorems in this section. The reader is referred to texts on linear algebra. Part (a) is not very difficult to prove, even in the case of general n. Part (b) cannot really be proved in general without a formal definition of determinant. However, the reader should verify (b) for 2×2 matrices by direct calculation.

We say that the square matrix $\mathcal{A}$ is **singular** if $\det(\mathcal{A}) = 0$. If $\det(\mathcal{A}) \neq 0$, we say that $\mathcal{A}$ is **nonsingular** or **invertible**.

Remark If $\mathcal{A}$ is a 3×3 matrix, then $\det(\mathcal{A})$ is the scalar triple product of the rows of $\mathcal{A}$, and its absolute value is the volume of the parallelepiped spanned by those rows. Therefore, $\mathcal{A}$ is nonsingular if and only if its rows span a parallelepiped of positive volume; the row vectors cannot all lie in the same plane. The same may be said of the columns of $\mathcal{A}$.

In general, an $n \times n$ matrix is singular if its rows (or columns), considered as vectors, satisfy one or more linear equations of the form

$$c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_n\mathbf{x}_n = \mathbf{0},$$

with at least one nonzero coefficient c_i. A set of vectors satisfying such a linear equation is called **linearly dependent** because one of the vectors can always be expressed as a linear combination of the others; if $c_1 \neq 0$, then

$$\mathbf{x}_1 = -\frac{c_2}{c_1}\mathbf{x}_2 - \frac{c_3}{c_1}\mathbf{x}_3 - \cdots - \frac{c_n}{c_1}\mathbf{x}_n.$$

All linear combinations of the vectors in a linearly dependent set of n vectors in $\mathbb{R}^n$ must lie in a **subspace** of dimension lower than n.

The $n \times n$ **identity matrix** is the matrix

$$\mathcal{I} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

with "1" in every position on the **main diagonal** and "0" in every other position. Evidently $\mathcal{I}$ commutes with every $n \times n$ matrix: $\mathcal{IA} = \mathcal{AI} = \mathcal{A}$. Also $\det(\mathcal{I}) = 1$. The identity matrix plays the same role in matrix algebra that the number 1 plays in arithmetic.

Any nonzero number x has a reciprocal x^{-1} such that $xx^{-1} = x^{-1}x = 1$. A similar situation holds for square matrices. The **inverse** of a *nonsingular* square matrix $\mathcal{A}$ is a nonsingular square matrix $\mathcal{A}^{-1}$ satisfying

$$\mathcal{AA}^{-1} = \mathcal{A}^{-1}\mathcal{A} = \mathcal{I}.$$

THEOREM 4

Every nonsingular square matrix $\mathcal{A}$ has a *unique* inverse $\mathcal{A}^{-1}$. Moreover, the inverse satisfies

(a) $\det(\mathcal{A}^{-1}) = \dfrac{1}{\det(\mathcal{A})}$,

(b) $(\mathcal{A}^{-1})^T = (\mathcal{A}^T)^{-1}$.

We will not have much cause to calculate inverses, but we note that it can be done by solving systems of linear equations, as the following simple example illustrates.

Example 5 Show that the matrix $\mathcal{A} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ is nonsingular and find its inverse.

Solution $\det(\mathcal{A}) = \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} = 1 + 1 = 2$. Therefore, $\mathcal{A}$ is nonsingular and invertible. Let $\mathcal{A}^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a-c & b-d \\ a+c & b+d \end{pmatrix},$$

so a, b, c, and d must satisfy the systems of equations

$$\begin{cases} a - c = 1 \\ a + c = 0 \end{cases} \qquad \begin{cases} b - d = 0 \\ b + d = 1. \end{cases}$$

Evidently $a = b = d = 1/2$, $c = -1/2$, and

$$\mathcal{A}^{-1} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

Generally, matrix inversion is not carried out by the method of the above example but rather by an orderly process of performing operations on the rows of the matrix to transform it into the identity. When the same operations are performed on the rows of the identity matrix, the inverse of the original matrix results. See a text on linear algebra for a description of the method. A singular matrix has no inverse.

Linear Transformations

A function $\mathbf{F}$ whose domain is the m-dimensional space $\mathbb{R}^m$ and whose range is contained in the n-dimensional space $\mathbb{R}^n$ is called a **linear transformation from $\mathbb{R}^m$ to $\mathbb{R}^n$** if it satisfies

$$\mathbf{F}(\lambda\mathbf{x} + \mu\mathbf{y}) = \lambda\mathbf{F}(\mathbf{x}) + \mu\mathbf{F}(\mathbf{y})$$

for all points $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^m$ and all real numbers λ and μ. To such a linear transformation F there corresponds an $n \times m$ matrix $\mathcal{F}$ such that for all $\mathbf{x}$ in $\mathbb{R}^m$,

$$\mathbf{F}(\mathbf{x}) = \mathcal{F}\mathbf{x},$$

or, expressed in terms of the components of $\mathbf{x}$,

$$\mathbf{F}(x_1, x_2, \cdots, x_m) = \mathcal{F} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix}.$$

We say that $\mathcal{F}$ is a **matrix representation** of the linear transformation $\mathbf{F}$. If $m = n$ so that $\mathbf{F}$ maps $\mathbb{R}^m$ into itself, then $\mathcal{F}$ is a square matrix. In this case $\mathcal{F}$ is nonsingular if and only if $\mathbf{F}$ is one-to-one and has the whole of $\mathbb{R}^m$ as range.

A composition of linear transformations is still a linear transformation and will have a matrix representation. The real motivation lying behind the definition of matrix multiplication is that the matrix representation of a *composition* of linear transformations is the *product* of the individual matrix representations of the transformations being composed.

THEOREM 5

If $\mathbf{F}$ is a linear transformation from $\mathbb{R}^m$ to $\mathbb{R}^n$ represented by the $n \times m$ matrix $\mathcal{F}$, and if $\mathbf{G}$ is a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^p$ represented by the $p \times n$ matrix $\mathcal{G}$, then the composition $\mathbf{G} \circ \mathbf{F}$ defined by

$$\mathbf{G} \circ \mathbf{F}(x_1, x_2, \ldots, x_m) = \mathbf{G}\Big(\mathbf{F}(x_1, x_2, \ldots, x_m)\Big)$$

is itself a linear transformation from $\mathbb{R}^m$ to $\mathbb{R}^p$ represented by the $p \times m$ matrix $\mathcal{G}\mathcal{F}$. That is,

$$\mathbf{G}\Big(\mathbf{F}(\mathbf{x})\Big) = \mathcal{G}\mathcal{F}\mathbf{x}.$$

Linear Equations

A system of n linear equations in n unknowns:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n \end{aligned}$$

can be written compactly as a single matrix equation,

$$\mathcal{A}\mathbf{x} = \mathbf{b},$$

where

$$\mathcal{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}, \qquad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}.$$

Compare the equation $\mathcal{A}\mathbf{x} = \mathbf{b}$ with the equation $ax = b$ for a single unknown x. The equation $ax = b$ has the unique solution $x = a^{-1}b$ provided $a \neq 0$. By analogy, the linear system $\mathcal{A}\mathbf{x} = \mathbf{b}$ has a unique solution given by

$$\mathbf{x} = \mathcal{A}^{-1}\mathbf{b},$$

provided $\mathcal{A}$ is nonsingular. To see this, just multiply both sides of the equation $\mathcal{A}\mathbf{x} = \mathbf{b}$ on the left by $\mathcal{A}^{-1}$; $\mathbf{x} = \mathcal{I}\mathbf{x} = \mathcal{A}^{-1}\mathcal{A}\mathbf{x} = \mathcal{A}^{-1}\mathbf{b}$.

If $\mathcal{A}$ is singular, then the system $\mathcal{A}\mathbf{x} = \mathbf{b}$ may or may not have a solution, and if a solution exists it will not be unique. Consider the case $\mathbf{b} = \mathbf{0}$ (the zero vector). Then any vector $\mathbf{x}$ perpendicular to all the rows of $\mathcal{A}$ will satisfy the system. Since the rows of $\mathcal{A}$ lie in a space of dimension less than n (because $\det(\mathcal{A}) = 0$), there will be at least a line of such vectors $\mathbf{x}$. Thus, solutions of $\mathcal{A}\mathbf{x} = \mathbf{0}$ are not unique if $\mathcal{A}$ is singular. The same must be true of the system $\mathcal{A}^T\mathbf{y} = \mathbf{0}$; there will be nonzero vectors $\mathbf{y}$ satisfying it if $\mathcal{A}$ is singular. But then, if the system $\mathcal{A}\mathbf{x} = \mathbf{b}$ has any solution $\mathbf{x}$, we must have

$$(\mathbf{y} \bullet \mathbf{b}) = \mathbf{y}^T\mathbf{b} = \mathbf{y}^T\mathcal{A}\mathbf{x} = (\mathbf{x}^T\mathcal{A}^T\mathbf{y})^T = (\mathbf{x}^T\mathbf{0})^T = (0).$$

Hence, $\mathcal{A}\mathbf{x} = \mathbf{b}$ can only have solutions for those vectors $\mathbf{b}$ that are perpendicular to *every* solution $\mathbf{y}$ of $\mathcal{A}^T\mathbf{y} = \mathbf{0}$.

A system of m linear equations in n unknowns may or may not have any solutions if $n < m$. It will have solutions if some $m - n$ of the equations are *linear combinations* (sums of multiples) of the other n equations. If $n > m$, then we can try to solve the m equations for m of the variables, allowing the solutions to depend on the other $n - m$ variables. Such a solution exists if the determinant of the coefficients of the m variables for which we want to solve is not zero. This is a special case of the **Implicit Function Theorem** which we will examine in Section 12.8.

Example 6 Solve $\begin{cases} 2x + y - 3z = 4 \\ x + 2y + 6z = 5 \end{cases}$ for x and y in terms of z.

Solution The system can be expressed in the form

$$\mathcal{A}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 + 3z \\ 5 - 6z \end{pmatrix} \qquad \text{where} \quad \mathcal{A} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

$\mathcal{A}$ has determinant 3 and inverse $\mathcal{A}^{-1} = \begin{pmatrix} 2/3 & -1/3 \\ -1/3 & 2/3 \end{pmatrix}$. Thus

$$\begin{pmatrix} x \\ y \end{pmatrix} = \mathcal{A}^{-1}\begin{pmatrix} 4 + 3z \\ 5 - 6z \end{pmatrix} = \begin{pmatrix} 2/3 & -1/3 \\ -1/3 & 2/3 \end{pmatrix}\begin{pmatrix} 4 + 3z \\ 5 - 6z \end{pmatrix} = \begin{pmatrix} 1 + 4z \\ 2 - 5z \end{pmatrix}$$

The solution is $x = 1 + 4z$, $y = 2 - 5z$. (Of course, this solution could have been found by elimination of x or y from the given equations without using matrix methods.) ■

The following theorem states a result of some theoretical importance expressing the solution of the system $\mathcal{A}\mathbf{x} = \mathbf{b}$ for nonsingular $\mathcal{A}$ in terms of determinants.

THEOREM 6

Cramer's Rule

Let $\mathcal{A}$ be a nonsingular $n \times n$ matrix. Then the solution $\mathbf{x}$ of the system

$$\mathcal{A}\mathbf{x} = \mathbf{b}$$

has components given by

$$x_1 = \frac{\det(\mathcal{A}_1)}{\det(\mathcal{A})}, \quad x_2 = \frac{\det(\mathcal{A}_2)}{\det(\mathcal{A})}, \quad \ldots, \quad x_n = \frac{\det(\mathcal{A}_n)}{\det(\mathcal{A})},$$

where $\mathcal{A}_j$ is the matrix $\mathcal{A}$ with its jth column replaced by the column vector $\mathbf{b}$. That is,

$$\det(\mathcal{A}_j) = \begin{vmatrix} a_{11} & \cdots & a_{1(j-1)} & b_1 & a_{1(j+1)} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2(j-1)} & b_2 & a_{2(j+1)} & \cdots & a_{2n} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & \cdots & a_{n(j-1)} & b_n & a_{n(j+1)} & \cdots & a_{nn} \end{vmatrix}.$$

The following example provides a concrete illustration of the use of Cramer's Rule to solve a specific linear system. However, Cramer's Rule is primarily used in a more general (theoretical) context; it is not efficient to use determinants to calculate solutions of linear systems.

Example 7 Find the point of intersection of the three planes

$$\begin{aligned} x + y + 2z &= 1 \\ 3x + 6y - z &= 0 \\ x - y - 4z &= 3. \end{aligned}$$

Solution The solution of the linear system above provides the coordinates of the intersection point. The determinant of the coefficient matrix of this system is

$$\det(\mathcal{A}) = \begin{vmatrix} 1 & 1 & 2 \\ 3 & 6 & -1 \\ 1 & -1 & -4 \end{vmatrix} = -32,$$

so the system does have a unique solution. We have

$$x = \frac{1}{-32}\begin{vmatrix} 1 & 1 & 2 \\ 0 & 6 & -1 \\ 3 & -1 & -4 \end{vmatrix} = \frac{-64}{-32} = 2,$$

$$y = \frac{1}{-32}\begin{vmatrix} 1 & 1 & 2 \\ 3 & 0 & -1 \\ 1 & 3 & -4 \end{vmatrix} = \frac{32}{-32} = -1,$$

$$z = \frac{1}{-32}\begin{vmatrix} 1 & 1 & 1 \\ 3 & 6 & 0 \\ 1 & -1 & 3 \end{vmatrix} = \frac{0}{-32} = 0.$$

The intersection point is $(2, -1, 0)$. ■

Quadratic Forms, Eigenvalues, and Eigenvectors

If $\mathbf{x}$ is a column vector in $\mathbb{R}^n$ and $\mathcal{A} = \big(a_{ij}\big)$ is an $n \times n$, real, symmetric matrix, (i.e., $a_{ij} = a_{ji}$ for $1 \le i, j \le n$), then the expression

$$Q(\mathbf{x}) = \mathbf{x}^T \mathcal{A}\mathbf{x} = \sum_{i,j=1}^{n} a_{ij} x_i x_j$$

is called a **quadratic form** on $\mathbb{R}^n$ corresponding to the matrix $\mathcal{A}$. Observe that $Q(\mathbf{x})$ is a real number for every n-vector $\mathbf{x}$.

We say that $\mathcal{A}$ is **positive definite** if $Q(\mathbf{x}) > 0$ for every nonzero vector $\mathbf{x}$. Similarly, $\mathcal{A}$ is **negative definite** if $Q(\mathbf{x}) < 0$ for every nonzero vector $\mathbf{x}$. We say that $\mathcal{A}$ is **positive semidefinite** (or **negative semidefinite**) if $Q(\mathbf{x}) \ge 0$ (or $Q(\mathbf{x}) \le 0$) for every nonzero vector $\mathbf{x}$.

If $Q(\mathbf{x}) > 0$ for some nonzero vectors $\mathbf{x}$ while $Q(\mathbf{x}) < 0$ for other such $\mathbf{x}$ (i.e., if $\mathcal{A}$ is neither positive semidefinite nor negative semidefinite), then we will say that $\mathcal{A}$ is **indefinite**.

Example 8 The expression $Q(x, y, z) = 3x^2 + 2y^2 + 5z^2 - 2xy + 4xz + 2yz$ is a quadratic form on $\mathbb{R}^3$ corresponding to the symmetric matrix

$$\mathcal{A} = \begin{pmatrix} 3 & -1 & 2 \\ -1 & 2 & 1 \\ 2 & 1 & 5 \end{pmatrix}.$$

Observe how the elements of the matrix are obtained from the coefficients of Q; the coefficients of x^2, y^2, and z^2 form the main diagonal elements, while the coefficients of the product terms are cut in half and half is put in each of the two corresponding symmetric off-diagonal positions.

The matrix $\mathcal{A}$ is positive definite since $Q(x, y, z)$ can be rewritten in the form

$$Q(x, y, z) = x^2 + (x - y)^2 + (x + 2z)^2 + (y + z)^2,$$

from which it is apparent that $Q(x, y, z) \geq 0$ for all (x, y, z) and $Q(x, y, z) = 0$ only if $x = y = z = 0$.

■

In Section 13.1 we will use the positive or negative definiteness of certain matrices to classify critical points of functions of several variables as local maxima and minima. Useful criteria for definiteness can be expressed in terms of the *eigenvalues* of the matrix $\mathcal{A}$.

We say that λ is an **eigenvalue** of the $n \times n$ square matrix $\mathcal{A} = (a_{ij})$ if there exists a *nonzero* column vector $\mathbf{x}$ such that $\mathcal{A}\mathbf{x} = \lambda\mathbf{x}$, or, equivalently,

$$(\mathcal{A} - \lambda\mathcal{I})\mathbf{x} = \mathbf{0},$$

where $\mathcal{I}$ is the $n \times n$ identity matrix. The nonzero vector $\mathbf{x}$ is called an **eigenvector** of $\mathcal{A}$ corresponding to the eigenvalue λ and can exist only if $\mathcal{A} - \lambda\mathcal{I}$ is a singular matrix, that is, if

$$\det(\mathcal{A} - \lambda\mathcal{I}) = \begin{vmatrix} a_{11} - \lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} - \lambda & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} - \lambda \end{vmatrix} = 0.$$

The eigenvalues of $\mathcal{A}$ must satisfy this nth-degree polynomial equation, so they can be either real or complex. The following theorems are proved in standard linear algebra texts.

THEOREM 7

If $\mathcal{A} = \left(a_{ij}\right)_{i,j=1}^{n}$ is a real, symmetric matrix, then

(a) all the eigenvalues of $\mathcal{A}$ are real,

(b) all the eigenvalues of $\mathcal{A}$ are nonzero if $\det(\mathcal{A}) \neq 0$,

(c) $\mathcal{A}$ is positive definite if all its eigenvalues are positive,

(d) $\mathcal{A}$ is negative definite if all its eigenvalues are negative,

(e) $\mathcal{A}$ is positive semidefinite if all its eigenvalues are nonnegative,

(f) $\mathcal{A}$ is negative semidefinite if all its eigenvalues are nonpositive,

(g) $\mathcal{A}$ is indefinite if it has at least one positive eigenvalue and at least one negative eigenvalue.

THEOREM 8

Let $\mathcal{A} = \left(a_{ij}\right)_{i,j=1}^{n}$ be a real symmetric matrix and consider the determinants

$$D_i = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1i} \\ a_{21} & a_{22} & \cdots & a_{2i} \\ \vdots & \vdots & \ddots & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ii} \end{vmatrix} \qquad \text{for } 1 \leq i \leq n.$$

Thus, $D_1 = a_{11}$, $D_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21} = a_{11}a_{22} - a_{12}^2$, etc.

(a) If $D_i > 0$ for $1 \le i \le n$, then $\mathcal{A}$ is positive definite.

(b) If $D_i > 0$ for even numbers i in $\{1, 2, \ldots, n\}$, and $D_i < 0$ for odd numbers i in $\{1, 2, \ldots, n\}$, then $\mathcal{A}$ is negative definite.

(c) If $\det(\mathcal{A}) = D_n \neq 0$ but neither of the above conditions hold, then $Q(\mathbf{x})$ is indefinite.

(d) If $\det(\mathcal{A}) = 0$, then $\mathcal{A}$ is not positive or negative definite and may be semidefinite or indefinite.

Example 9 For the matrix $\mathcal{A}$ of Example 8, we have

$$D_1 = 3 > 0, \quad D_2 = \begin{vmatrix} 3 & -1 \\ -1 & 2 \end{vmatrix} = 5 > 0, \quad D_3 = \begin{vmatrix} 3 & -1 & 2 \\ -1 & 2 & 1 \\ 2 & 1 & 5 \end{vmatrix} = 10 > 0,$$

which reconfirms that the quadratic form of that exercise is positive definite. ■

Exercises 10.6

Evaluate the matrix products in Exercises 1–4.

1. $\begin{pmatrix} 3 & 0 & -2 \\ 1 & 1 & 2 \\ -1 & 1 & -1 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 3 & 0 \\ 0 & -2 \end{pmatrix}$

2. $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$

3. $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} w & x \\ y & z \end{pmatrix}$

4. $\begin{pmatrix} w & x \\ y & z \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

5. Evaluate $\mathcal{A}\mathcal{A}^T$ and $\mathcal{A}^2 = \mathcal{A}\mathcal{A}$, where

$$\mathcal{A} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

6. Evaluate $\mathbf{x}\mathbf{x}^T$, $\mathbf{x}^T\mathbf{x}$, and $\mathbf{x}^T\mathcal{A}\mathbf{x}$, where

$$\mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad \text{and} \quad \mathcal{A} = \begin{pmatrix} a & p & q \\ p & b & r \\ q & r & c \end{pmatrix}.$$

Evaluate the determinants in Exercises 7–8.

7. $\begin{vmatrix} 2 & 3 & -1 & 0 \\ 4 & 0 & 2 & 1 \\ 1 & 0 & -1 & 1 \\ -2 & 0 & 0 & 1 \end{vmatrix}$

8. $\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ -2 & 0 & 2 & 4 \\ 3 & -3 & 2 & -2 \end{vmatrix}$

9. Show that if $\mathcal{A} = (a_{ij})$ is an $n \times n$ matrix for which $a_{ij} = 0$ whenever $i > j$, then $\det(\mathcal{A}) = \prod_{k=1}^{n} a_{kk}$, the product of the elements on the main diagonal of $\mathcal{A}$.

10. Show that $\begin{vmatrix} 1 & 1 \\ x & y \end{vmatrix} = y - x$, and

$$\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix} = (y - x)(z - x)(z - y).$$

Try to generalize this result to the $n \times n$ case.

11. Verify the associative law $(\mathcal{A}\mathcal{B})\mathcal{C} = \mathcal{A}(\mathcal{B}\mathcal{C})$ by direct calculation for three arbitrary 2×2 matrices.

12. Show that $\det(\mathcal{A}^T) = \det(\mathcal{A})$ for $n \times n$ matrices by induction on n. Start with the 2×2 case.

13. Verify by direct calculation that $\det(\mathcal{A}\mathcal{B}) = \det(\mathcal{A})\det(\mathcal{B})$ holds for two arbitrary 2×2 matrices.

14. Let $\mathcal{A}_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$. Show that $(\mathcal{A}_\theta)^T = (\mathcal{A}_\theta)^{-1} = \mathcal{A}_{-\theta}$.

Find the inverses of the matrices in Exercises 15–16.

15. $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$

16. $\begin{pmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 2 & 1 & 3 \end{pmatrix}$

17. Use your result from Exercise16 to solve the linear system

$$\begin{cases} x - z = -2 \\ -x + y = 1 \\ 2x + y + 3z = 13. \end{cases}$$

18. Solve the system of Exercise 17 by using Cramer's Rule.

19. Solve the system $\begin{cases} x_1 + x_2 + x_3 + x_4 = 0 \\ x_1 + x_2 + x_3 - x_4 = 4 \\ x_1 + x_2 - x_3 - x_4 = 6 \\ x_1 - x_2 - x_3 - x_4 = 2. \end{cases}$

20. Verify Theorem 5 for the special case where F and G are linear transformations from $\mathbb{R}^2$ to $\mathbb{R}^2$.

In Exercises 21–26, classify the given symmetric matrices as positive or negative definite, positive or negative semidefinite, or indefinite.

21. $\begin{pmatrix} -1 & 1 \\ 1 & -2 \end{pmatrix}$

22. $\begin{pmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

23. $\begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$

24. $\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

25. $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix}$

26. $\begin{pmatrix} 2 & 0 & 1 \\ 0 & 4 & -1 \\ 1 & -1 & 1 \end{pmatrix}$

10.7 Using Maple for Vector and Matrix Calculations

The use of a computer algebra system can free us from much of the tedious calculation needed to do calculus. This is especially true of calculations in multivariable or vector calculus, where the calculations can quickly become unmanageable as the number of variables increases. This author's colleague, Dr. Robert Israel, has written an excellent book, *Calculus, the Maple Way*, to show how Maple can be used effectively for doing calculus involving both single-variable and multivariable functions.

In this book we will occasionally call on the power of Maple to carry out calculations involving functions of several variables and vector-valued functions of one or more variables. This section illustrates some of the most basic techniques. The examples here were calculated using Maple V Release 5, but Maple 6 gives almost identical output.

Most of Maple's capability to deal with vectors and matrices is not in its kernel but is written into a package of procedures called **linalg**. Therefore, it is customary to load this package at the beginning of a session where it will be needed:

```
> with(linalg):
```

One usually completes a Maple command with a semicolon rather than a colon. You can use a colon to suppress output. In this case the suppression of output is not complete; there are one or two warning messages printed, but they can safely be ignored. Had we used a semicolon to complete the command the result would have also included a list of all the procedures defined in the linalg package.

Maple 6 (and later releases) include a second linear algebra package called **LinearAlgebra**. It is in some respects superior to linalg, especially for heavy-duty numerical calculations using large matrices. It is also a little easier to use. For our purposes, however, the older linalg package is sufficient and has two advantages over the LinearAlgebra package: it defines certain procedures for differentiating vector functions that we will use later, and it is included in earlier versions of Maple. To simplify its use we will add a few new definitions as described below.

Vectors

First we create a few definitions of procedures to simplify our handling of vectors. These are to be found in the file **vecops.def** available on the website `www.pearsoned.ca/text/adams_calc`. This file can be read in at the beginning of a Maple session with the command

```
> read "vecops.def";
```

Reading in vecops.def will not only activate the new definitions, but it will also load the linalg package, so that does not have to be done separately. Here is a listing of vecops.def:

```
print('Loading the linalg package');
with(linalg):
evl := V -> simplify(convert(evalm(V),list));
'&.' := (U,V) -> sum(U[i]*V[i],i=1..vectdim(U));
len := V -> sqrt(V &.  V);
unitv := V -> evl((1/len(V))*V);
'&x' := (U,V) -> if vectdim(U)=3 and vectdim(V)=3 then
[U[2]*V[3]-U[3]*V[2],U[3]*V[1]-U[1]*V[3],
U[1]*V[2]-U[2]*V[1]]
else print('Error - vectors must have dimension 3');
RETURN('')
fi:
```

Maple is capable of dealing with vectors as simple lists of elements; for example, `U := [-5,2,x];` defines U as the 3-vector with components -5, 2, and x. These components can be referenced separately using indices placed in square brackets. In this case, the input

```
> U[1]+U[3];
```

will produce the output $-5 + x$.

Vector addition and scalar multiplication are represented, just as their analogues for numbers, by using "+" and "*". Sometimes Maple may not automatically simplify the result into a single vector. In this case we can apply the `evl()` function (defined in **vecops.def**) to the result to force the simplification. `evl(%)` does three things: it expresses the previous result as a 1×3 matrix, then converts the matrix to a list, then simplifies any elements of the list.

```
> U := [2,-2,1]; V := [a, b, c];
```

$$U := [2, -2, 1]$$

$$V := [a, b, c]$$

```
> U + V; 2*U + x*V;
```

$$[a + 2, b - 2, c + 1]$$

$$[4, -4, 2] + x\ [a, b, c]$$

```
> evl(%);
```

$$[4 + x\ a, -4 + x\ b, 2 + x\ c]$$

In Release 5 of Maple V and in Maple 6 the "%" symbol refers to the result of the previous calculation.

The linalg package provides definitions of the dot product and cross product of two vectors that can be called by the functions `dotprod(U,V);` and `crossprod(U,V);` In **vecops.def** we have defined two binary operators, "`&.`" and "`&x`" to simplify the calculation of dot and cross products by providing a notation similar to the one we use when writing mathematics. Thus, we calculate the dot and cross products of U and V as `U &. V` and `U &x V`. It is not necessary to leave spaces around these operators except when `&x` is followed by a letter (like V); then leaving a space is necessary so Maple won't think you are using a variable by the name of `xV`.

```
> U &. V;   U &x V;
```

$$2\,a - 2\,b + c$$

$$[-2\,c - b, a - 2\,c, 2\,b + 2\,a]$$

If you want the cross product of two plane vectors, you must give the third components as zero. An attempt to calculate a cross product of vectors that are not 3-dimensional will generate an error message and produce an empty string.

```
> [1,1,0] &x [2,-1,0];
```

$$[0, 0, -3]$$

```
> N := [1,1] &x [2,-1]; N;
```

Error - vectors must have dimension 3

$$N :=$$

It is useful to have functions for the length of a vector and for a unit vector in the direction of a given vector. **vecops.def** provides functions `len()` and `unitv()` for these purposes.

```
> len([2,1,-2]); unitv([2,1,-2]);
```

$$3$$

$$\left[\frac{2}{3}, \frac{1}{3}, \frac{-2}{3}\right]$$

The plane through $(2, 1, -1)$ perpendicular to the line of intersection of the two planes $2x + 3y + z = 5$ and $3x - 2y - 4z = 1$ has normal **N** given by

```
> N := [2,3,1] &x [3,-2,-4];
```

$$N := [-10, 11, -13]$$

Thus, the plane has equation

```
> N &. ([x,y,z] - [2,1,-1]) = 0;
```

$$-4 - 10 * x + 11 * y - 13 * z = 0$$

or, as we would write it, $10x - 11y + 13z = -4$.

Now let us use Maple to verify the identity

$$(\mathbf{U}\times\mathbf{V})\times\mathbf{W} = (\mathbf{U} \bullet \mathbf{W})\mathbf{V} - (\mathbf{W} \bullet \mathbf{V})\mathbf{U}.$$

First, we define **U**, **V**, and **W** to be vectors with arbitrary components:

```
> U := vector(3); V := vector(3); W := vector(3);
```

$$U := array(1..3, [])$$

$$V := array(1..3, [])$$

$$W := array(1..3, [])$$

This output confirms that each vector is an array of 3 elements. Now we calculate the left-hand side minus the right-hand side of the identity:

```
> (U &x V) &x W - (U &. W)*V + (W &. V)*U;
```

We expected to get the result [0, 0, 0] confirming that the identity is true. However, what we got was a long vector of algebraic expressions involving the components of the three vectors. It needs simplification. Entering `evl(%)` (i.e., using `evl` to simplify the previous result) produces the desired [0, 0, 0].

Finally, we observe that all of the functions and operators defined in **vecops.def** with the exception of the cross product, `&x`, can be applied to vectors of any dimension:

```
> W := [seq(i,i=1..10)];   W &. W;   len(W);
```

$$W := [1, 2, 3, 4, 5, 6, 7, 8, 9, 10]$$

$$385$$

$$\sqrt{385}.$$

Matrices

We do not need the extra vector operations in `vecops.def` to calculate with matrices in Maple, but we do need many of the procedures defined in the linalg package, so we must load this package if it is not already loaded.

```
> with(linalg):
```

In Maple, a matrix can be defined by feeding a list of lists to the function `matrix()`. An $m \times n$ matrix (with m rows and n columns) corresponds to a list of m elements (a column vector), each of which is a list of n numbers or expressions (the rows of the matrix). For example, we specify the 2×3 matrix M with rows $[1, 1, 1]$ and $[2, 1, 3]$ by

```
> M := matrix([[1,1,1],[2,1,3]]);
```

$$M := \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 3 \end{bmatrix}$$

Alternatively, we can specify the number of rows and columns of the matrix and then provide the elements in order in a single list:

```
> M := matrix(2,3,[1,1,1,2,1,3]);
```

$$M := \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 3 \end{bmatrix}$$

Individual elements of a matrix can be addressed using two indices, the first indicating the row and the second the column:

```
> M[2,1]; M[2,3];
```

$$2$$

$$3$$

The transpose T of the matrix M is the 3×2 matrix whose rows are the columns of M. It is calculated using the `transpose()` function:

```
> T := transpose(M);
```

$$T := \begin{bmatrix} 1 & 2 \\ 1 & 1 \\ 1 & 3 \end{bmatrix}$$

The product, AB, of two matrices A and B is calculated using the binary operator `&*`; that is, we calculate `A &* B`. Of course, the number of columns of A must be equal to the number of rows of B. The resulting matrix product will be left in symbolic form (i.e., as A `&*` B) unless we force its evaluation by using the matrix evaluation function `evalm()`. Here, we calculate the product of M and its transpose T and then evaluate it with `evalm`. The result is a 2×2 matrix:

```
> P := M &* T; evalm(P);
```

$$P := M\& * T$$

$$\begin{bmatrix} 3 & 6 \\ 6 & 14 \end{bmatrix}$$

On the other hand, `T &* M` is a 3×3 matrix:

```
> Q := evalm(T &* M);
```

$$Q := \begin{bmatrix} 5 & 3 & 7 \\ 3 & 2 & 4 \\ 7 & 4 & 10 \end{bmatrix}$$

Observe that both `M &* T` and `T &* M` are symmetric, square matrices. (This is always true of the matrix product of a real matrix and its transpose.)

The determinant of a square matrix (one with equal numbers of rows and columns) is given by the `det()` function:

```
> A := matrix([[1,1,1],[2,1,3],[5,-1,-2]]); det(A);
```

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 3 \\ 5 & -1 & -2 \end{bmatrix}$$

$$13$$

The inverse of a nonsingular square matrix A can be calculated as `inverse(A)`:

```
> Ainv := inverse(A);
```

$$Ainv := \begin{bmatrix} 1/13 & 1/13 & 2/13 \\ 19/13 & -7/13 & -1/13 \\ -7/13 & 6/13 & -1/13 \end{bmatrix}$$

```
> evalm(A &* Ainv);
```

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

When a matrix is multiplied on the right by a list, Maple treats the list as a column vector. When a matrix is multiplied on the left by a list, Maple treats the list as a row vector. In either case, the result can be simplified by `evalm()`:

```
> evalm(A &* [x,y,z]); evalm([x,y,z] &* A);
```

$$[x + y + z, 2x + y + 3z, 5x - y - 2z]$$

$$[x + 2y + 5z, x + y - z, x + 3y - 2z]$$

A set of n linear equations in n variables can be written in the form $A\mathbf{X} = \mathbf{B}$, where A is an $n \times n$ matrix and $\mathbf{X}$ and $\mathbf{B}$ are column n-vectors. Thus, the solution can be calculated as $\mathbf{X} = A^{-1}\mathbf{B}$. For example, the system

$$x + y + z = 2, \qquad 2x + y + 3z = 9, \qquad 5x - y - 2z = 1$$

has the matrix A defined above as its coefficient matrix, and $\mathbf{B} = [2, 9, 1]$. The solution of the system is:

```
> X := evalm(Ainv &* [2,9,1]);
```

$$X := [1, -2, 3]$$

that is, $x = 1$, $y = -2$, $z = 3$. Maple provides a simpler way of solving the system $A\mathbf{X} = \mathbf{B}$; we just need to use the function `linsolve(A,B)`:

```
> X := linsolve(A,[2,9,1]);
```

$$X := [1, -2, 3]$$

`linsolve` is better at solving linear systems than is matrix inversion, since it can solve some systems for which the matrix is singular. Consider the two systems

$$\begin{aligned} x + y &= 1 \\ 2x + 2y &= 2 \end{aligned} \quad \text{and} \quad \begin{aligned} x + y &= 1 \\ 2x + 2y &= 1 \end{aligned}$$

The first system has a one-parameter family of solutions $x = 1 - t$, $y = t$ for arbitrary t (which Maple V calls $t[1]$ and Maple VI calls $\_t_1$ to allow for the possibility that some systems can have more than one such arbitrary constant in its solution). The second system has no solutions.

```
> L := matrix([[1,1],[2,2]]); C1 := [1,2]; C2 := [1,1];
```

$$L := \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$$

$$C1 := [1, 2]$$

$$C2 := [1, 1]$$

```
> X := linsolve(L,C1);
```

$$X := [1 - t[1], t[1]]$$

```
> X := linsolve(L,C2);
```

$$X :=$$

Now `X` is undefined, indicating no solutions for $L\mathbf{x} = \mathbf{C}2$. Since the matrix `inverse(L)` does not exist, not even the solution of the first system, $L\mathbf{x} = \mathbf{C}1$, can be found as `X := inverse(L) &* C1`.

```
> X := inverse(L) &* C1;
Error, (in inverse) singular matrix
```

You may wonder why Cramer's Rule is not used to calculate solutions of linear systems. The reason is that it is very inefficient, requiring many more operations than the elimination method used by `linsolve`.

The linalg package has procedures for finding the eigenvalues and eigenvectors of matrices. For a real symmetric matrix, the eigenvalues are always real.

```
> K := matrix([[3,1,-1],[1,4,1],[-1,1,3]]);
```

$$K := \begin{bmatrix} 3 & 1 & -1 \\ 1 & 4 & 1 \\ -1 & 1 & 3 \end{bmatrix}$$

```
> eigenvals(K); evalf(%);
```

$$4, \quad 3 + \sqrt{3}, \quad 3 - \sqrt{3}$$

$$4., \quad 4.732050808, \quad 1.267949192$$

Since all three eigenvalues are positive, K is a positive definite matrix. Our main use for eigenvalues will be the classification of critical points of functions of several variables. This use does not require knowledge of the corresponding eigenvectors, but if we did need to know them, we could have used the function `eigenvectors(K)` instead. The output of `eigenvectors(A)` is a set of lists, each of which contains an eigenvalue of A, the multiplicity of that eigenvalue (i.e., the dimension of the corresponding eigenspace), and a set of linearly independent eigenvectors of A corresponding to that eigenvalue and forming a basis of the corresponding eigenspace.

Exercises 10.7

Use Maple to calculate the quantities in Exercises 1–2.

1. The distance between the line through $(3, 0, 2)$ parallel to the vector $2\mathbf{i}+\mathbf{j}-2\mathbf{k}$ and the line through $(1, 2, 4)$ parallel to $\mathbf{i}+3\mathbf{j}+4\mathbf{k}$

2. The angle between the vector $\mathbf{i}-\mathbf{j}+2\mathbf{k}$ and the plane through the origin containing the vectors $\mathbf{i}-2\mathbf{j}-3\mathbf{k}$ and $2\mathbf{i}+3\mathbf{j}+4\mathbf{k}$

Use Maple to verify the identities in Exercises 3–4

3. $\mathbf{U}\bullet(\mathbf{V}\times\mathbf{W})=\mathbf{V}\bullet(\mathbf{W}\times\mathbf{U})=\mathbf{W}\bullet(\mathbf{U}\times\mathbf{V})$

4. $(\mathbf{U}\times\mathbf{V})\times(\mathbf{U}\times\mathbf{W})=(\mathbf{U}\bullet(\mathbf{V}\times\mathbf{W}))\mathbf{U}$

In Exercises 5–10 define Maple functions to produce the indicated results. You may use functions already defined in **vecops.def**.

5. A function `sp(U,V)` that gives the scalar projection of vector **U** along the nonzero vector **V**

6. A function `vp(U,V)` that gives the vector projection of vector **U** along the nonzero vector **V**

7. A function `ang(U,V)` that gives the angle between the nonzero vectors **U** and **V** in degrees as a decimal number

8. A function `unitn(U,V)` that gives a unit vector normal to the two nonparallel vectors **U** and **V** in 3-space

9. A function `VolT(U,V,W)` that gives the volume of the tetrahedron in 3-space that is spanned by the vectors **U**, **V**, and **W**

10. A function `dist(A,B)` giving the distance between two points having position vectors **A** and **B**. Use your function to find the distance between $[1, 1, 1, 1]$ and $[3, -1, 2, 5]$

In Exercises 11–12, use `linsolve` to solve the given systems.

11. $$\begin{cases} u+2v+3x+4y+5z=20 \\ 6u-v+6x+2y-3z=0 \\ 2u+8v-8x-2y+z=6 \\ u+v+x+y+z=5 \\ 10u-3v+3x-2y+2z=5 \end{cases}$$

12. $$\begin{cases} u+v+x+y+z=6 \\ u-2v+3x-4y+5x=-5 \\ u-v+2x-2y+5z=-1 \\ 2u-3v+5x-6y+8z=-6 \\ 2u-v+4x-3y+6z=1 \end{cases}$$

13. Evaluate the determinant of the coefficient matrix for the system in Exercise 11.

14. Find the eigenvalues of the coefficient matrix for the system in Exercise 12. Quote your answers as decimal numbers (use `evalf`) to 5 decimal places. Do you think any of them are really complex?

15. Find the inverse of the matrix

$$A=\begin{bmatrix} 1 & 1/2 & 1/3 \\ 1/2 & 1/3 & 1/4 \\ 1/3 & 1/4 & 1/5 \end{bmatrix}.$$

16. Find, in decimal form (using `evalf(eigenvals(A))`) the eigenvalues of the matrix A of the previous exercise and the eigenvalues of its inverse. How do you account for the fact that some of the eigenvalues appear to be complex?

Chapter Review

Key Ideas

- **What is each of the following?**
  - ◇ a neighbourhood ◇ an open set ◇ a closed set
  - ◇ the boundary of a set ◇ the interior of a set
  - ◇ a vector in 3-space ◇ the dot product of vectors
  - ◇ the cross product of two vectors in $\mathbb{R}^3$
  - ◇ a scalar triple product ◇ a vector triple product
  - ◇ a matrix ◇ a determinant

◇ a plane ◇ a straight line ◇ a cone
◇ a cylinder ◇ an ellipsoid ◇ a paraboloid
◇ a hyperboloid of 1 sheet ◇ a hyperboloid of 2 sheets
◇ the transpose of a matrix ◇ the inverse of a matrix
◇ a linear transformation ◇ an eigenvalue of a matrix

- **What is the angle between the vectors u and v?**
- **How do you calculate u×v, given the components of u and v?**
- **What is an equation of the plane through P_0 having normal vector N?**
- **What is an equation of the straight line through P_0 parallel to a?**
- **Given two 3×3 matrices A and B, how do you calculate AB?**
- **What is the distance from P_0 to the plane $Ax + By + Cz + D = 0$?**
- **What is Cramer's Rule and how is it used?**

Review Exercises

Describe the sets of points in 3-space that satisfy the given equations or inequalities in Exercises 1–18.

1. $x + 3z = 3$
2. $y - z \ge 1$
3. $x + y + z \ge 0$
4. $x - 2y - 4z = 8$
5. $y = 1 + x^2 + z^2$
6. $y = z^2$
7. $x = y^2 - z^2$
8. $z = xy$
9. $x^2 + y^2 + 4z^2 < 4$
10. $x^2 + y^2 - 4z^2 = 4$
11. $x^2 - y^2 - 4z^2 = 0$
12. $x^2 - y^2 - 4z^2 = 4$

* 13. $(x - z)^2 + y^2 = 1$

* 14. $(x - z)^2 + y^2 = z^2$

15. $\begin{cases} x + 2y = 0 \\ z = 3 \end{cases}$
16. $\begin{cases} x + y + 2z = 1 \\ x + y + z = 0 \end{cases}$
17. $\begin{cases} x^2 + y^2 + z^2 = 4 \\ x + y + z = 3 \end{cases}$
18. $\begin{cases} x^2 + z^2 \le 1 \\ x - y \ge 0 \end{cases}$

Find equations of the planes and lines specified in Exercises 19–28.

19. The plane through the origin perpendicular to the line
$$\frac{x-1}{2} = \frac{y+3}{-1} = \frac{z+2}{3}$$
20. The plane through $(2, -1, 1)$ and $(1, 0, -1)$ parallel to the line in Exercise 19
21. The plane through $(2, -1, 1)$ perpendicular to the planes $x - y + z = 0$ and $2x + y - 3z = 2$
22. The plane through $(-1, 1, 0)$, $(0, 4, -1)$, and $(2, 0, 0)$
23. The plane containing the line of intersection of the planes $x + y + z = 0$ and $2x + y - 3z = 2$, and passing through the point $(2, 0, 1)$
24. The plane containing the line of intersection of the planes $x + y + z = 0$ and $2x + y - 3z = 2$, and perpendicular to the plane $x - 2y - 5z = 17$
25. The vector parametric equation of the line through $(2, 1, -1)$ and $(-1, 0, 1)$
26. Standard form equations of the line through $(1, 0, -1)$ parallel to each of the planes $x - y = 3$ and $x + 2y + z = 1$
27. Scalar parametric equations of the line through the origin perpendicular to the plane $3x - 2y + 4z = 5$
28. The vector parametric equation of the line that joins points on the two lines
$$\mathbf{r} = (1 + t)\mathbf{i} - t\mathbf{j} - (2 + 2t)\mathbf{k}$$
$$\mathbf{r} = 2t\mathbf{i} + (t - 2)\mathbf{j} - (1 + 3t)\mathbf{k}$$
and is perpendicular to both those lines

Express the given conditions or quantities in Exercises 29–30 in terms of dot and cross products.

29. The three points with position vectors $\mathbf{r}_1$, $\mathbf{r}_2$, and $\mathbf{r}_3$ all lie on a straight line.
30. The four points with position vectors $\mathbf{r}_1$, $\mathbf{r}_2$, $\mathbf{r}_3$, and $\mathbf{r}_4$ do not all lie on a plane.
31. Find the area of the triangle with vertices $(1, 2, 1)$, $(4, -1, 1)$, and $(3, 4, -2)$.
32. Find the volume of the tetrahedron with vertices $(1, 2, 1)$, $(4, -1, 1)$, $(3, 4, -2)$, and $(2, 2, 2)$.
33. Show that the matrix
$$\mathcal{A} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 \\ 4 & 3 & 2 & 1 \end{pmatrix}$$
has an inverse, and find the inverse $\mathcal{A}^{-1}$.
34. Let $\mathcal{A} = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \\ 1 & 0 & -1 \end{pmatrix}$. What condition must the vector $\mathbf{b}$ satisfy in order that the equation $\mathcal{A}\mathbf{x} = \mathbf{b}$ has solutions $\mathbf{x}$? What are the solutions $\mathbf{x}$ if $\mathbf{b}$ satisfies the condition?
35. Is the matrix $\begin{pmatrix} 3 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 2 \end{pmatrix}$ positive or negative definite or neither?

Challenging Problems

1. Show that the distance d from point P to the line AB can be expressed in terms of the position vectors of P, A, and B by
$$d = \frac{|(\mathbf{r}_A - \mathbf{r}_P) \times (\mathbf{r}_B - \mathbf{r}_P)|}{|\mathbf{r}_A - \mathbf{r}_B|}$$
2. For any vectors $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$, and $\mathbf{x}$, show that
$$\begin{aligned}(\mathbf{u} \times \mathbf{v}) \times (\mathbf{w} \times \mathbf{x}) &= \big((\mathbf{u} \times \mathbf{v}) \bullet \mathbf{x}\big)\mathbf{w} - \big((\mathbf{u} \times \mathbf{v}) \bullet \mathbf{w}\big)\mathbf{x} \\ &= \big((\mathbf{w} \times \mathbf{x}) \bullet \mathbf{u}\big)\mathbf{v} - \big((\mathbf{w} \times \mathbf{x}) \bullet \mathbf{v}\big)\mathbf{u}.\end{aligned}$$

In particular, show that

$$(\mathbf{u} \times \mathbf{v}) \times (\mathbf{u} \times \mathbf{w}) = \big((\mathbf{u} \times \mathbf{v}) \bullet \mathbf{w}\big)\mathbf{u}.$$

3. Show that the area A of a triangle with vertices $(x_1, y_1, 0)$, $(x_2, y_2, 0)$, and $(x_3, y_3, 0)$, in the xy-plane is given by

$$A = \frac{1}{2}\left|\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}\right|.$$

4. (a) If L_1 and L_2 are two **skew** (i.e., nonparallel and non-intersecting) lines, show that there is a pair of parallel planes P_1 and P_2 such that L_1 lies in P_1 and L_2 lies in P_2.

(b) Find parallel planes containing the following two lines: L_1 through points $(1, 1, 0)$ and $(2, 0, 1)$, and L_2 through points $(0, 1, 1)$ and $(1, 2, 2)$.

5. What condition must the vectors $\mathbf{a}$ and $\mathbf{b}$ satisfy to ensure that the equation $\mathbf{a} \times \mathbf{x} = \mathbf{b}$ has solutions? If this condition is satisfied, find all solutions of the equation. Describe the set of solutions.