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Calculus: theory, examples, exercises

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Методическое пособие разработано в рамках курса Calculus для студентов первого курса МИЭФ ГУ-ВШЭ. Значительная часть материалов было подготовлено автором в рамках проекта Национального фонда подготовки кадров "Развитие учебной программы МИЭФ ГУ-ВШЭ международного уровня по направлению “Экономика”.

Методическое пособие основано на более чем десятилетнем опыте преподавания данного курса в МИЭФ.

Руководство подготовлено на английском языке, поскольку данный курс изучается в МИЭФ в рамках программы подготовки студентов к экзаменам в формате APT для поступления на Внешнюю программу Лондонского университета и преподается на английском языке.

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Chapter 1.
PRE-CALCULUS

1.1 Domain and range

Definition. A **function** f is a rule by which values of the **independent** variable x are assigned values of the **dependent** variable y:

$$y = f(x)$$

In calculus, functions can be defined by using a *table*, a *graph* or a *formula*.

Definition. The set of numbers x at which $f(x)$ is defined is the **domain** of f.

Definition. The **range** of f is the set of values which are assumed by f on its domain.

Find the domain of the following functions:

1.1. $y = \dfrac{x^2}{4+x}$

1.2. $y = \sqrt{4x - x^2}$

1.3. $y = \dfrac{1}{\sqrt{4x-x^2}}$

1.4. $y = \ln(x+2) + \ln(x-2)$

1.5. $y = \dfrac{1}{\ln(1-x)} + \sqrt{x+2}$

1.6. $y = \sin^{-1}\left(\dfrac{x}{4}\right)$

1.7. $y = \cos^{-1}\left(\dfrac{2}{2+\sin x}\right)$

1.8. $y = \sin^{-1}\left(\dfrac{2x}{1+x}\right)$

1.9. $y = \log\left(\sin\dfrac{\pi}{x}\right)$

1.10. $y = \ln(\ln x)$

Are the following functions the same?

1.11. $f(x) = x$, $g(x) = \sqrt{x^2}$

1.12. $f(x) = \ln x^2$, $g(x) = 2\ln x$

1.13. $f(x) = 2^{\log_2 x}$, $g(x) = x$

1.14. $f(x) = \tan x \cot x$, $g(x) = 1$

1.15. $f(x) = \tan x$, $g(x) = \dfrac{1}{\cot x}$

1.16. $f(x) = \cos(\cos^{-1} x)$, $g(x) = \cos^{-1}(\cos x)$

Find the domain and range of the following functions.

1.17. $y = \sqrt{2+x-x^2}$

1.18. $y = \log(1-2\cos x)$

1.19. $y = \cos^{-1} \dfrac{2x}{1+x^2}$

1.20. $y = \sin^{-1}\left(\log \dfrac{x}{10}\right)$

1.2 Function graphs

Without using the derivative, sketch the following functions.

1.21. $y = (x-1)^2$

1.22. $y = x^2+2x-4$

1.23. $y = 5+6x-x^2$

1.24. $y = \sqrt{1-x^2}$

1.25. $y = |x|+x$

1.26. $y = (|x|+2)(|x|-3)$

1.27. $y = \big|3-x\big|-3$

1.28. $y = \dfrac{1}{1+x^2}$

1.29. $y = \log|x|$

1.30. $y = \big|3^x-1\big|$

1.31. $y = 2^{x+1}+2^{x+2}+2^{x+3}$

1.32. $y = \left(\dfrac{1}{2}\right)^{|x|}$

1.33. $y = \sin|x|$

1.34. $y = |\sin x|$

1.35. $y = \sqrt{2x}$

1.36. $y = (x+1)^{1/3}$

1.37. $y = e^{x/2}$

1.38. $y = e^{x-x^2}$

1.39. $y = 1+0.5^x$

1.40. $y = \big|\ln(x+1)\big|$

1.41. $y = \tan \dfrac{x}{2}$

1.42. $y = \cos\left(x+\dfrac{\pi}{4}\right)$

1.43. $y = x+\dfrac{1}{x}$

1.44. $y = \dfrac{1}{x^2+x+1}$

1.45. $y = \dfrac{1}{x^2-1}$

1.46. $y = \dfrac{1-x}{x+1}$

1.47. $y = 6\cos x+8\sin x$

1.48. $y = e^x\cos x$

1.49. $y = e^{-x^2}\cos(2x)$

1.50. $y = \sin^2 x$

1.51. $y = x^{1/\ln x}$

1.52. $y = \dfrac{1}{e^x\cos x}$

1.53. $y = \dfrac{|x-3|+|x+1|}{|x+3|+|x+1|}$

1.54. $y = |\cos x|+|\sin x|$

1.55. $y = \sin x|\cos x|+\cos x|\sin x|$

1.56. $y = \sin\left(2\sin^{-1} x\right)$

1.57. $y = \log_{1/2}\left(x - \dfrac{1}{2}\right) + \log_2 \sqrt{4x^2 - 4x + 1}$

1.58. Compare the graphs of $f(x) = \sin\left(\cos^{-1} x\right)$ and $g(x) = \sqrt{1 - x^2}$. What is the relation between these functions?

1.59. Compare the graphs of $f(x) = \cos\left(\tan^{-1} x\right)$ and $g(x) = \frac{1}{\sqrt{1+x^2}}$. What is the relation between these functions?

1.60. Compare the graphs of $f(x) = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$ and $g(x) = 2\tan^{-1} x$ for $x \geq 0$. What is the relation between these functions?

1.61. Prove that if the graph of $f(x)$, $x \in \mathbb{R}$ is symmetric with respect to the vertical lines $x = a$ and $x = b$, $a \neq b$, then f is periodic.

1.62. A function is *antiperiodic* if there exists some number $T > 0$ such that $f(x+T) = -f(x)$. Prove that an antiperiodic function is periodic with a period of $2T$.

1.63. Can a function defined for all x be both even and odd? If so, find all such functions.

1.64. Let $f(x)$ be the sum of two periodic functions. Is it periodic?

Chapter 2.
SEQUENCES

Definition. The **natural numbers** or **positive integers** are the numbers $1, 2, 3, \ldots$ A **sequence** of real numbers is an assignment of a real number to each natural number.

Examples of sequences are arithmetic and geometric progressions, e.g. $a_n = a_1 + (n-1)d$ and $b_n = b_1 q^{n-1}$.

Definition. Let x_n be a sequence of real numbers. The number A is the **limit** of this sequence if for *any* positive number ε there exists a number N such that $|x_n - A| < \varepsilon$ for any $n > N$.

The limit of a sequence is denoted $\lim\limits_{n\to\infty} x_n = a$.

Example 2.1. Using the definition of the limit of a sequence, show that

$$\text{a) } \lim_{n\to\infty} \frac{4n}{2n+1} = 2; \qquad \text{b) } \lim_{n\to\infty} \frac{2^n+1}{2^n} = 1.$$

Solution. a) By the definition,

$$\lim_{n\to\infty} \frac{4n}{2n+1} = 2 \quad \Leftrightarrow \quad \forall \varepsilon > 0 \quad \exists N, \quad \left| \frac{4n}{2n+1} - 2 \right| < \varepsilon \quad \forall n > N.$$

Simplifying,

$$\left| \frac{4n - 4n - 2}{2n+1} \right| = \left| \frac{-2}{2n+1} \right| = \frac{2}{2n+1} < \varepsilon \quad \Rightarrow \quad 2n\varepsilon + \varepsilon > 2 \quad \Rightarrow \quad n > \frac{2-\varepsilon}{2\varepsilon}.$$

Therefore, $N = \left[\frac{2-\varepsilon}{2\varepsilon}\right]$ (the greatest whole number that does not exceed $\frac{2-\varepsilon}{2\varepsilon}$). For any $\varepsilon > 0$ the value of the sequence terms $\left(\frac{4n}{2n+1}\right)$ for any $n > N$ will differ from the limit of the sequence by less than ε.

b) By the definition,

$$\lim_{n\to\infty} \frac{2^n+1}{2^n} = 1 \quad \Leftrightarrow \quad \forall \varepsilon > 0 \quad \exists N, \quad \left| \frac{2^n+1}{2^n} - 1 \right| < \varepsilon \quad \forall n > N.$$

Simplifying,

$$\left| \frac{2^n + 1 - 2^n}{2^n} \right| = \left| \frac{1}{2^n} \right| = \frac{1}{2^n} < \varepsilon \quad \Rightarrow \quad 2^n > \frac{1}{\varepsilon} \quad \Rightarrow \quad n > \log_2 \frac{1}{\varepsilon}.$$

Therefore, $N = \left[\log_2 \frac{1}{\varepsilon}\right]$.

Example 2.2. Using the definition of the limit of a sequence, show that

$$\lim_{n\to\infty} \frac{3n^3 + 4n^2 - 5n + 6}{n^3 + 4n^2 + 6n + 20} = 3.$$

Solution. By the definition,

$$\lim_{n\to\infty}\frac{3n^3+4n^2-5n+6}{n^3+4n^2+6n+20}=3 \quad\Leftrightarrow$$

$$\Leftrightarrow \forall\varepsilon>0 \quad \exists N, \quad \left|\frac{3n^3+4n^2-5n+6}{n^3+4n^2+6n+20}-3\right|<\varepsilon \quad \forall n>N.$$

Simplifying,

$$\left|\frac{-8n^2-23n-54}{n^3+4n^2+6n+20}\right|<\varepsilon$$

The difference between this inequality and the similar inequalities in the previous example is that this one cannot be solved exactly. However, according to the definition, it is only necessary to show that the values of the sequence terms differ from the sequence limit by less than ε *starting at some point*; it is not at all necessary to find the exact place where this starts to happen:

$$\left|\frac{-8n^2-23n-54}{n^3+4n^2+6n+20}\right|=\frac{8n^2+23n+54}{n^3+4n^2+6n+20}<\frac{8n^2+23n^2+54n^2}{n^3}=\frac{85}{n}$$

Therefore, if $\dfrac{85}{n}<\varepsilon$, then $\left|\dfrac{-8n^2-23n-54}{n^3+4n^2+6n+20}\right|<\varepsilon$; $N=\left[\dfrac{85}{\varepsilon}\right]$.

Prove the following limits using the definition of the limit of a sequence.

2.1. $\lim\limits_{n\to\infty}\dfrac{3n-1}{5n+1}=\dfrac{3}{5}$

2.2. $\lim\limits_{n\to\infty}\dfrac{n}{n^2+1}=0$

2.3. $\lim\limits_{n\to\infty}\dfrac{n^2+2n}{n^2+2}=1$

2.4. $\lim\limits_{n\to\infty}\dfrac{2^n+4}{3^n-5}=0$

2.5. $\lim\limits_{n\to\infty}\dfrac{3n^2-1}{n^2+n+1}=3$

2.6. $\lim\limits_{n\to\infty}\dfrac{n^3-10n^2+5n-6}{2n^3+4n^2-n+1}=0.5$

2.7. $\lim\limits_{n\to\infty}2^{2-n}=0$

2.8. $\lim\limits_{n\to\infty}\dfrac{\sin n!\sqrt[5]{n^3}}{n+1}=0$

2.9. $\lim\limits_{n\to\infty}\dfrac{\sin n+\cos n}{\sqrt{n}}=0$

Definition. A sequence is **infinitely small** as $x\to a$ if $\lim\limits_{x\to a}x_n=0$.

Definition. A sequence is **infinitely large** as $x\to a$ if for any number M, however large, there is a number N such that $|x_n|>M$ for any $n>N$. This is concisely written as follows: $\lim\limits_{n\to\infty}|x_n|=\infty$.

Most of the time, we need to consider infinitely large sequences such that $\lim\limits_{n\to\infty}x_n=\infty$. For most other common infinitely large sequences, $\lim\limits_{n\to\infty}x_n=-\infty$.

Important: Infinitely large sequences do not have a limit; in other words, the limit of an infinitely large sequence does not exist. The expression $\lim\limits_{n\to\infty}x_n=\infty$ is a convention *only,* which shows the manner in which the limit does not exist.

Theorem. If the sequence $\{x_n\}$ is infinitely large, then the sequence $\left\{\frac{1}{x_n}\right\}$ is infinitely small, and vice versa.

Properties of sequence limits

Assuming that all the limits given below exist,

1. $\lim\limits_{n\to\infty} kx_n = k \lim\limits_{n\to\infty} x_n$, where k is a constant;
2. $\lim\limits_{n\to\infty} (x_n + y_n) = \lim\limits_{n\to\infty} x_n + \lim\limits_{n\to\infty} y_n$;
3. $\lim\limits_{n\to\infty} x_n y_n = \lim\limits_{n\to\infty} x_n \lim\limits_{n\to\infty} y_n$;
4. $\lim\limits_{n\to\infty} \dfrac{x_n}{y_n} = \dfrac{\lim\limits_{n\to\infty} x_n}{\lim\limits_{n\to\infty} y_n}$, $\lim\limits_{n\to\infty} y_n \neq 0$;
5. If $f(x)$ is an elementary function, then $\lim\limits_{n\to\infty} f(x_n) = f\left(\lim\limits_{n\to\infty} x_n\right)$.

Example 2.3. Find the following limits.

$$\text{a) } \lim_{n\to\infty} \frac{2n+3}{2n+4}; \quad \text{b) } \lim_{n\to\infty} \frac{n^3+n^2+n+1}{3n^3-2n^2+n-1}; \quad \text{c) } \lim_{n\to\infty} \left(\sqrt{n^2+1}-n\right).$$

Solution. a) Dividing the numerator and denominator by n,

$$\lim_{n\to\infty} \frac{2n+3}{2n+4} = \lim_{n\to\infty} \frac{2+\frac{3}{n}}{2+\frac{4}{n}} = \frac{2+0}{2+0} = 1.$$

b) Dividing the numerator and denominator by n^3,

$$\lim_{n\to\infty} \frac{n^3+n^2+n+1}{3n^3-2n^2+n-1} = \lim_{n\to\infty} \frac{1+\frac{1}{n}+\frac{1}{n^2}+\frac{1}{n^3}}{3-2\frac{1}{n}+\frac{1}{n^2}-\frac{1}{n^3}} = \frac{1+0+0+0}{3+0+0+0} = \frac{1}{3}.$$

c) Multiplying by the complement,

$$\lim_{n\to\infty} \left(\sqrt{n^2+1}-n\right) = \lim_{n\to\infty} \frac{\left(\sqrt{n^2+1}-n\right)\left(\sqrt{n^2+1}+n\right)}{\sqrt{n^2+1}+n} =$$

$$= \lim_{n\to\infty} \frac{n^2+1-n^2}{\sqrt{n^2+1}+n} = \lim_{n\to\infty} \frac{1}{\sqrt{n^2+1}+n} = \left[\frac{1}{\infty}\right] = 0.$$

In Example 2.3, it was necessary to transform the expressions before finding their limits. This is because a *direct* application of the properties of sequence limits could not be used. For instance, in a) and b) the numerator and denominator were infinitely large, making a direct application of property 4 impossible. In c), the items in the difference were also infinitely large, making a direct application of property 2 impossible. These expressions are called **indeterminacies**: in a) and b), the indeterminacy was of the type $\left(\frac{\infty}{\infty}\right)$, while in c) it was of the type $(\infty-\infty)$.

Another common indeterminacy is the type (1^∞). The most common case is:

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e,$$

which is the definition of the number e, the base of the natural logarithm. It is easy to show that

$$\lim_{n\to\infty}\left(1-\frac{1}{n}\right)^n = \frac{1}{e}.$$

Example 2.4. Find the following limits.

$$\text{a) } \lim_{n\to\infty}\left(\frac{n+2}{n}\right)^n; \quad \text{b) } \lim_{n\to\infty}\left(\frac{n^2-n+1}{n^2+n+1}\right)^{2n}.$$

Solution. a) We have

$$\lim_{n\to\infty}\left(\frac{n+2}{n}\right)^n = \lim_{n\to\infty}\left(1+\frac{2}{n}\right)^{\frac{n}{2}\cdot 2} = e^2,$$

because (using a change of variables $m=\frac{n}{2}$)

$$\lim_{n\to\infty}\left(1+\frac{2}{n}\right)^{\frac{n}{2}} = \lim_{m\to\infty}\left(1+\frac{1}{m}\right)^m = e.$$

b) In the same way,

$$\lim_{n\to\infty}\left(\frac{n^2-n+1}{n^2+n+1}\right)^{2n} = \lim_{n\to\infty}\left(1-\frac{2n}{n^2+n+1}\right)^{\frac{n^2+n+1}{2n}\cdot\frac{2n}{n^2+n+1}\cdot 2n} =$$

$$= \frac{1}{e^{\lim\limits_{n\to\infty}\frac{4n^2}{n^2+n+1}}} = e^{-4}.$$

Find the following limits.

2.10. $\lim\limits_{n\to\infty}\frac{(n+1)^2}{2n^2}$

2.11. $\lim\limits_{n\to\infty}\frac{2n^3-500n^2+19}{220n^2+50n}$

2.12. $\lim\limits_{n\to\infty}\frac{n(n+1)(n+2)}{(2n+3)(3n+4)(4n+5)}$

2.13. $\lim\limits_{n\to\infty}\frac{2356n^3+6n^2}{0.007n^4-9n^3+61}$

2.14. $\lim\limits_{n\to\infty}\frac{(2n+1)^3-(n-1)^3}{(2n+1)^3+(n-1)^3}$

2.15. $\lim\limits_{n\to\infty}\frac{4n^3-3n^2+2n-5}{5n^3+8n-17}.$

2.16. $\lim\limits_{n\to\infty}\left(\frac{4n^3+n^2-8}{3n^3+n+20}\right)^3$

2.17. $\lim\limits_{n\to\infty}\frac{\sqrt{n^2+2}+\sqrt{n}}{\sqrt{n^2-2}+\sqrt{2n^2+1}}$

2.18. $\lim\limits_{n\to\infty} \dfrac{\sin(n^2+n)}{n+2}$

2.19. $\lim\limits_{n\to\infty} \dfrac{\log_2 n}{\log_3 n}$

2.20. $\lim\limits_{n\to\infty} \dfrac{1+2+3+\ldots+n}{n^2}$

2.21. $\lim\limits_{n\to\infty} \dfrac{n^3}{1^2+2^2+3^2+\ldots+n^2}$

2.22. $\lim\limits_{n\to\infty} \dfrac{1+a+a^2+\ldots+a^n}{1+b+b^2+\ldots+b^n}$, $(|a|<1, |b|<1)$

2.23. $\lim\limits_{n\to\infty} \left(\dfrac{1+2+3+\ldots+n}{n+2} - \dfrac{n}{2}\right)$

2.24. $\lim\limits_{n\to\infty} \left(n\left(\dfrac{1}{n^3}+\dfrac{3}{n^3}+\ldots+\dfrac{2n-1}{n^3}\right)\right)$

2.25. $\lim\limits_{n\to\infty} 2^{\frac{4n-5}{8n+5}}$

2.26. $\lim\limits_{n\to\infty} \dfrac{\sqrt[3]{n^3+2n^2+2n-1}}{n-2}$

2.27. $\lim\limits_{n\to\infty} \dfrac{\sqrt[3]{n^2+5n+10}}{3n-1}$

2.28. $\lim\limits_{n\to\infty} \dfrac{\left(\sqrt{n^2+1}+n\right)^2}{\sqrt[3]{n^6+1}}$

2.29. $\lim\limits_{n\to\infty} \left(\dfrac{n^2+4}{n\sin n+n^2} \cdot \dfrac{1}{n^3+1}\right)$

2.30. $\lim\limits_{n\to\infty} \left(\sqrt{4n^2+n}-2n\right)$

2.31. $\lim\limits_{n\to\infty} n\left(\sqrt{n^2+1}-n\right)$

2.32. $\lim\limits_{n\to\infty} \left(\sqrt{n^2+n}-n\right)$

2.33. $\lim\limits_{n\to\infty} \left(\sqrt{3n^2+2n}-\sqrt{3n^2-4n}\right)$

2.34. $\lim\limits_{n\to\infty} \left(\sqrt{n+2\sqrt{n}}-\sqrt{n}\right)$

2.35. $\lim\limits_{n\to\infty} \left(\dfrac{n+1}{n-1}\right)^n$

2.36. $\lim\limits_{n\to\infty} \left(\dfrac{n^2+n+1}{n^2+2}\right)^{3n}$

2.37. $\lim\limits_{n\to\infty} \left(\dfrac{n+3}{n+4}\right)^{2-4n}$

2.38. $\lim\limits_{n\to\infty} \left(\dfrac{3\sqrt{n}-1}{3\sqrt{n}}\right)^{\sqrt{n-1}}$

2.39. Consider two sequences, x_n and y_n. If x_n converges, and y_n does not, what can be said about the convergence of the sequences x_n+y_n and $x_n y_n$?

2.40. Consider two sequences, x_n and y_n, neither of which is convergent. What can be said about the convergence of the sequences x_n+y_n and $x_n y_n$?

2.41. Let $\lim\limits_{n\to\infty} x_n y_n = 0$. Is it true that either $\lim\limits_{n\to\infty} x_n = 0$ or $\lim\limits_{n\to\infty} y_n = 0$, or both?

2.42. If $\lim\limits_{n\to\infty} |a_n| = |a|$, is it true that $\lim\limits_{n\to\infty} a_n = a$? Is it true that a_n converges?

2.43. Let b_n be some sequence. Is it true that if $\lim\limits_{n\to\infty} a_n = 0$, then $\lim\limits_{n\to\infty} a_n b_n = 0$?

2.44. Give an example of diverging sequences a_n and b_n such that a_n+b_n converges.

2.45. Give an example of diverging sequences a_n and b_n such that $a_n b_n$ converges.

2.46. Give an example of diverging sequences a_n and b_n such that the sequences a_n+b_n and $a_n b_n$ both converge.

2.47. Consider the sequence defined by the series of equations

$$3 = \frac{2}{x_1} = x_1 + \frac{2}{x_2} = x_2 + \frac{2}{x_3} = x_3 + \frac{2}{x_4} = \ldots$$

Guess at a general expression for x_n in terms of n and prove your formula by induction. Then find $\lim\limits_{n\to\infty} x_n$.

Chapter 3.
FUNCTION LIMITS

3.1 Algebraic methods for finding limits

Definition. The number A is the **limit of the function f at $x = a$** if for *any* positive number ε there exists a positive number δ such that $|f(x) - A| < \varepsilon$ for any $0 < |x - a| < \delta$.

The limit of a function at a point is denoted $\lim_{x\to a} f(x) = A$.

Definition. The number A is the **limit of the function f at infinity** if for *any* positive number ε there exists a number $x_0 > 0$ such that $|f(x) - A| < \varepsilon$ for any $x > x_0$.

The limit of a function at infinity is denoted $\lim_{x\to\infty} f(x) = A$.

Definition. The number A is the **limit of the function f at minus infinity** if for *any* positive number ε there exists a number $x_0 > 0$ such that $|f(x) - A| < \varepsilon$ for any $x < -x_0$.

The limit of a function at minus infinity is denoted $\lim_{x\to-\infty} f(x) = A$.

The limit of a function at infinity can be found using the same methods as for the limit of a sequence. The limit of a function at minus infinity can also be found using these methods, but remembering that the argument of the function (x) is negative.

Properties of function limits

Assuming that all the limits given below exist,

1. $\lim_{x\to a} kf(x) = k \lim_{x\to a} f(x)$, where k is a constant;
2. $\lim_{x\to a} \big(f(x) + g(x)\big) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$;
3. $\lim_{x\to a} f(x)g(x) = \lim_{x\to a} f(x) \lim_{x\to a} g(x)$;
4. $\lim_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$, $\lim_{x\to a} g(x) \neq 0$;
5. If $f(x)$ is an elementary function, then $\lim_{x\to a} f\big(g(x)\big) = f\Big(\lim_{x\to a} g(x)\Big)$.

Definition. An **elementary function** is a function built from a *finite* number of constants and power, exponential, logarithmic, trigonometric and inverse trigonometric functions through composition and combinations using the four elementary operations (addition, subtraction, multiplication and division).

Special limits

1. $\lim\limits_{x\to 0}\dfrac{\sin x}{x}=1\,;$

2. $\lim\limits_{x\to\infty}\left(1+\dfrac{1}{x}\right)^{x}=\lim\limits_{x\to 0}(1+x)^{1/x}=e\,.$

Theorem (The sandwich theorem)

If there exists an interval $(a-\varepsilon,a+\varepsilon)$ such that $g(x)\le f(x)\le h(x)$ for all $x\in(a-\varepsilon,a+\varepsilon)$, and $\lim\limits_{x\to a}g(x)=\lim\limits_{x\to a}h(x)=A$, then $\lim\limits_{x\to a}f(x)=A$.

Example 3.1. Find the following limits:

$$\text{a) }\lim_{x\to 3}\frac{x+2}{x-1};\quad \text{b) }\lim_{x\to 1}\frac{x^2-6x+5}{x^2+3x-4};\quad \text{c) }\lim_{x\to -1}\frac{x+1}{\sqrt{5+x}-2}.$$

Solution.

a) This limit can be found directly:

$$\lim_{x\to 3}\frac{x+2}{x-1}=\frac{3+2}{3-1}=\frac{5}{2}.$$

b) Factor the numerator and denominator:

$$\lim_{x\to 1}\frac{x^2-6x+5}{x^2+3x-4}=\lim_{x\to 1}\frac{(x-5)(x-1)}{(x+4)(x-1)}=\lim_{x\to 1}\frac{x-5}{x+4}=-\frac{4}{5}.$$

c) Multiplying the numerator and denominator by the complement,

$$\lim_{x\to -1}\frac{x+1}{\sqrt{5+x}-2}=\lim_{x\to -1}\frac{(x+1)\left(\sqrt{5+x}+2\right)}{\left(\sqrt{5+x}-2\right)\left(\sqrt{5+x}+2\right)}=$$

$$=\lim_{x\to -1}\frac{(x+1)\left(\sqrt{5+x}+2\right)}{5+x-4}=\lim_{x\to -1}\left(\sqrt{5+x}+2\right)=4.$$

Example 3.2. Find $\lim\limits_{x\to\infty}\left(1-\dfrac{1}{x}\right)^{x}$.

Solution. Limits of this type can be reduced to the second special limit:

$$\lim_{x\to\infty}\left(1-\frac{1}{x}\right)^{x}=\lim_{x\to\infty}\left(\frac{x-1}{x}\right)^{x}=\lim_{x\to\infty}\left(\frac{x}{x-1}\right)^{-x}=\lim_{x\to\infty}\left(1+\frac{1}{x-1}\right)^{-x}=$$

$$=\lim_{x\to\infty}\left(\left(1+\frac{1}{x-1}\right)^{x-1}\right)^{\frac{-x}{x-1}}=e^{-\lim\limits_{x\to\infty}\frac{x}{x-1}}=e^{-1}.$$

Example 3.3. Find $\lim\limits_{x\to 0} x\sin\frac{1}{x}$.

Solution. The difficulty here is the presence of $\sin\frac{1}{x}$, which does not have a limit as $x \to 0$. However, $-1 \le \sin\frac{1}{x} \le 1$, so we have

$$-|x| \le x\sin\frac{1}{x} \le |x|.$$

Since $\lim\limits_{x\to 0}(-|x|) = 0$ and $\lim\limits_{x\to 0}|x| = 0$, we conclude on the basis of the sandwich theorem that $\lim\limits_{x\to 0} x\sin\frac{1}{x} = 0$.

Find the following limits.

3.1. $\lim\limits_{x\to\infty}\frac{x}{x+1}$

3.2. $\lim\limits_{x\to\infty}\frac{3x-1}{4x+1}$

3.3. $\lim\limits_{x\to\infty}\frac{(2x+1)^2}{4x^2}$

3.4. $\lim\limits_{x\to\infty}\left(\frac{x^3}{x^2+1}-x\right)$

3.5. $\lim\limits_{x\to 0}\frac{x^3-3x+1}{x-4}$

3.6. $\lim\limits_{x\to -2}\frac{x^2-4}{x+2}$

3.7. $\lim\limits_{x\to 0}\frac{x^2-1}{2x^2-x-1}$.

3.8. $\lim\limits_{x\to 1}\frac{x^2-1}{2x^2-x-1}$

3.9. $\lim\limits_{x\to\infty}\frac{x^2-1}{2x^2-x-1}$

3.10. $\lim\limits_{x\to -2}\frac{x^3+3x^2+2x}{x^2-x-6}$

3.11. $\lim\limits_{x\to 2}\frac{x^2-4x+4}{x^3-4x}$

3.12. $\lim\limits_{x\to -1/3}\frac{27x^3+1}{3x^2-2x-1}$

3.13. $\lim\limits_{x\to 1}\frac{x^4-2x+1}{x^6-2x+1}$

3.14. $\lim\limits_{x\to 1}\frac{x^3-4x+3}{x^4-5x+4}$

3.15. $\lim\limits_{x\to 0}\frac{(1+2x)(2+3x)(3+4x)-6}{x}$

3.16. $\lim\limits_{x\to\infty}\frac{(x+1)(x+2)(x+3)}{(2x-1)^3}$

3.17. $\lim\limits_{x\to\infty}\frac{(x+10)^5(6x-20)^5}{(3x+1)^{10}}$

3.18. $\lim\limits_{x\to -1}\frac{\left(x^3+5x^2+7x+3\right)^5}{\left(x^2+3x+2\right)^{10}}$

3.19. $\lim\limits_{x\to\frac{1}{2}}\frac{8x^3-1}{6x^2-5x+1}$

3.20. $\lim\limits_{x\to -1}\frac{x^3-2x-1}{x^5-2x-1}$

3.21. $\lim\limits_{x\to 1}\frac{x^{60}-1}{x^{30}-1}$

3.22. $\lim\limits_{x\to 1}\frac{(x-1)\cos\pi x}{x^3-1}$

3.23. $\lim\limits_{x\to 1}\frac{\sqrt{x}-1}{x-1}$

3.24. $\lim\limits_{x\to 1}\frac{x^2-\sqrt{x}}{\sqrt{x}-1}$

3.25. $\lim\limits_{x\to 2}\frac{\sqrt{x+2}-2}{x-2}$

3.26. $\lim\limits_{x\to 0}\frac{\sqrt{x^2+4x+9}+x-3}{25x^2+5x}$

3.27. $\lim\limits_{x\to 4}\frac{\sqrt{3x-3}-3}{2-\sqrt{x}}$

3.28. $\lim\limits_{x\to 5}\frac{\sqrt{x+11}-\sqrt{21-x}}{x^2-25}$

3.29. $\lim\limits_{x\to 81} \dfrac{\sqrt[4]{x}-3}{\sqrt{x}-9}$

3.30. $\lim\limits_{x\to 0} \dfrac{\sqrt[3]{1-x+2x^2}-1}{2x+x^2}$

3.31. $\lim\limits_{x\to 0} \dfrac{x^2}{\sqrt[5]{1+5x}-(1+x)}$

3.32. $\lim\limits_{x\to 0} \dfrac{\sqrt{1+x}-1}{\sqrt[3]{1+x}-1}$

3.33. $\lim\limits_{x\to 1} \left(\dfrac{3}{1-\sqrt{x}}-\dfrac{2}{1-\sqrt[3]{x}}\right)$

3.34. $\lim\limits_{x\to 7} \dfrac{\sqrt{x+2}-\sqrt[3]{x+20}}{\sqrt[4]{x+9}-2}$

3.35. $\lim\limits_{x\to 4} \dfrac{\sqrt{x}-2\sqrt[3]{x-3}}{3-\sqrt[4]{85-x}}$

3.36. $\lim\limits_{x\to 1} \dfrac{1-\sqrt[10]{x}}{1-\sqrt[15]{x}}$

3.37. $\lim\limits_{x\to 1} \dfrac{\sqrt[m]{x}-1}{\sqrt[n]{x}-1}$, $\{n,m\}\in\mathbb{N}$

3.38. $\lim\limits_{x\to\infty} \left(\sqrt{x^2+1}-x\right)$

3.39. $\lim\limits_{x\to\infty} \left(\sqrt{x^2+x}-x\right)$

3.40. $\lim\limits_{x\to\infty} \left(\sqrt{x^2+3x}-x\right)$

3.41. $\lim\limits_{x\to\infty} \left(x-\sqrt{x^2-4x}\right)$

3.42. $\lim\limits_{x\to\infty} \left(\sqrt{x+\sqrt{x}}-\sqrt{x}\right)$

3.43. $\lim\limits_{x\to\infty} \left(\sqrt[3]{x+\left(x+\sqrt[3]{x}\right)^{2/3}}-\sqrt[3]{x}\right)$

3.44. $\lim\limits_{x\to\infty} \sqrt{x}\left(\sqrt{x+1}+\sqrt{x+2}-2\sqrt{x}\right)$

3.45. $\lim\limits_{x\to\infty} \left(1+\dfrac{2}{x}\right)^x$

3.46. $\lim\limits_{x\to\infty} \left(\dfrac{2x+1}{2x-1}\right)^x$

3.47. $\lim\limits_{x\to 0} \left(\dfrac{1+2x}{1-2x}\right)^{1/x}$

3.48. $\lim\limits_{x\to\infty} \left(\dfrac{x+1}{x}\right)^x$

3.49. $\lim\limits_{x\to\infty} \left(\dfrac{x}{x+1}\right)^x$

3.50. $\lim\limits_{x\to\infty} \left(\dfrac{x+1}{x-1}\right)^x$

3.51. $\lim\limits_{x\to\infty} \left(\dfrac{x^2+2x+2}{x^2+2x+1}\right)^{3x^2-1}$

3.52. $\lim\limits_{x\to\infty} \left(\dfrac{x+2}{2x+1}\right)^{x^2}$

3.53. $\lim\limits_{x\to\infty} \left(\dfrac{x^2+1}{x^2-2}\right)^{x^2}$

3.54. $\lim\limits_{x\to \frac{\pi}{3}} \dfrac{\sin(x-\pi/3)}{1-2\cos x}$

3.55. $\lim\limits_{x\to 0} x^2\sin\dfrac{2+x}{x}$

3.56. $\lim\limits_{x\to 0} \sin x\sin\dfrac{1}{x}$

3.57. If $\lim\limits_{x\to\infty} f(x)$ exists, and $\lim\limits_{x\to\infty} g(x)$ does not, what can be said about the limits $\lim\limits_{x\to\infty} (f(x)+g(x))$ and $\lim\limits_{x\to\infty} (f(x)g(x))$?

3.58. If the limits $\lim\limits_{x\to\infty} f(x)$ and $\lim\limits_{x\to\infty} g(x)$ do not exist, is it true that the limits $\lim\limits_{x\to\infty} (f(x)+g(x))$ and $\lim\limits_{x\to\infty} (f(x)g(x))$ do not exist as well?

3.59. Let $\lim\limits_{x\to\infty} (f(x)g(x))=0$. Is it true that either $\lim\limits_{x\to\infty} f(x)=0$ or $\lim\limits_{x\to\infty} g(x)=0$, or both?

3.2 One-sided limits

Definition. The **left limit** of f at $x = a$ equals A if for *any* positive number ε there exists a positive number δ such that $|f(x) - A| < \varepsilon$ for any $a - \delta < x < a$.

The left limit is denoted $\lim\limits_{x \to a^-} f(x) = A$ or $\lim\limits_{x \to a-0} f(x) = A$.

Definition. The **right limit** of f at $x = a$ equals A if for *any* positive number ε there exists a positive number δ such that $|f(x) - A| < \varepsilon$ for any $a < x < a + \delta$.

The right limit is denoted $\lim\limits_{x \to a^+} f(x) = A$ or $\lim\limits_{x \to a+0} f(x) = A$.

Theorem. The following statements are equivalent: $\lim\limits_{x \to a} f(x) = A$ and $\lim\limits_{x \to a^-} f(x) = \lim\limits_{x \to a^+} f(x) = A$.

In other words, the limit of a function at $x = a$ exists if and only if the left limit equals the right limit.

Example 3.4. Find the left and right limits of the function

$$f(x) = \begin{cases} x^2, & 0 \le x \le 1, \\ 3 - x, & 1 < x \le 2 \end{cases}$$

at the point $x = 1$.

Solution. $\lim\limits_{x \to 1^-} f(x) = \lim\limits_{x \to 1^-} x^2 = 1$; $\quad \lim\limits_{x \to 1^+} f(x) = \lim\limits_{x \to 1^+} (3 - x) = 2$.

Example 3.5. Find the left and right limits of the function

$$f(x) = \frac{x}{x - 3}$$

at the point $x = 3$.

Solution. $\lim\limits_{x \to 3^-} f(x) = \lim\limits_{x \to 3^-} \dfrac{x}{x - 3} = \left(\dfrac{3}{-0}\right) = -\infty$;

$\lim\limits_{x \to 3^+} f(x) = \lim\limits_{x \to 3^+} \dfrac{x}{x - 3} = \left(\dfrac{3}{0}\right) = \infty$;

Find the left and right limits at the given point.

3.60. $f(x) = \dfrac{x}{|x|}$ at $x = 0$

3.61. $f(x) = \dfrac{1}{x - 2}$ at $x = 2$

3.62. $f(x) = \dfrac{1}{(x + 2)^2}$ at $x = -2$

3.63. $f(x) = \dfrac{x^2 + x - 6}{|x - 2|}$ at $x = 2$

3.64. $f(x) = \begin{cases} 3 - 2x, & x \le 1, \\ 3x - 5, & x > 1 \end{cases}$ at $x = 1$

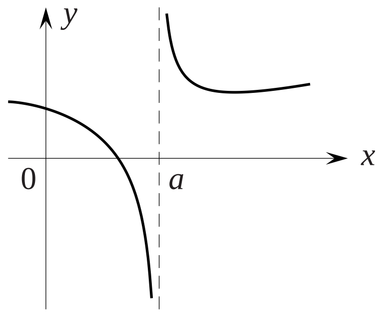
3.3 Function asymptotes

Definition. A straight line is an **asymptote** of a function, if the distance between a point on the graph of the function and that line approaches zero as the point approaches infinity.

There are two kinds of asymptotes:

I. Vertical asymptotes

Definition. The line $x = a$ is a **vertical asymptote** of $f(x)$, if either $\lim\limits_{x\to a^-} f(x)$ or $\lim\limits_{x\to a^+} f(x)$ equals ∞ or $-\infty$.

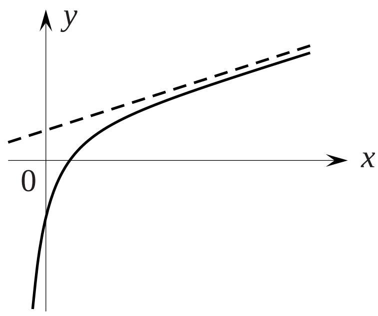


II.Slant asymptotes

Definition. Slant asymptotes are lines that the function graph approaches at $x \to \pm\infty$. Their equation is $y = kx + b$, where

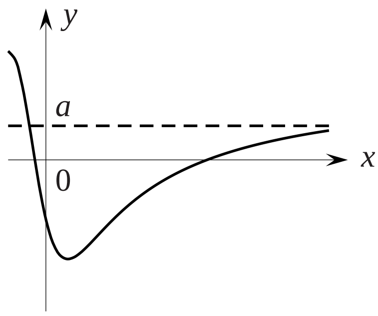
$$k = \lim_{x\to\pm\infty} \frac{f(x)}{x}, \quad b = \lim_{x\to\pm\infty} (f(x) - kx).$$

Slant asymptotes at $x \to \infty$ and $x \to -\infty$ should be considered separately.



Definition. The line $y = a$ is a **horizontal asymptote** of $f(x)$, if $\lim\limits_{x\to\infty} f(x) = a$ or $\lim\limits_{x\to-\infty} f(x) = a$.

Horizontal asymptotes are a special case of slant asymptotes, when $k = 0$.



Example 3.6. Find the asymptotes of the function $f(x) = \dfrac{x^2 - 2x + 2}{x - 3}$.

Solution. First we will consider whether or not this function has vertical asymptotes, i.e. points where either $\lim\limits_{x\to a^+} f(x)$ or $\lim\limits_{x\to a^-} f(x)$ is infinity. The only points where the function can be infinitely large is where the denominator of the function becomes zero: $x = 3$. Indeed, we have

$$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} \frac{x^2 - 2x + 2}{x - 3} = \left(\frac{5}{-0}\right) = -\infty,$$

so $x = 3$ is indeed a vertical asymptote of $f(x)$. (It would also be possible to consider the right limit of $f(x)$ at $x = 3$, which is ∞.)

Next, consider the behaviour of $f(x)$ as $x \to \infty$:

$$k = \lim_{x\to\infty} \frac{x^2 - 2x + 2}{x(x-3)} = 1;$$

$$b = \lim_{x\to\infty} \left(\frac{x^2 - 2x + 2}{x-3} - x \right) = \lim_{x\to\infty} \frac{x+2}{x-3} = 1.$$

Therefore, the asymptote of $f(x)$ at ∞ is $y = x + 1$.

It can be noted that the calculations given above will be exactly the same if $x \to -\infty$; therefore the asymptote of $f(x)$ as $x \to -\infty$ will be the same: $y = x + 1$.

Example 3.7. Find the asymptotes of the function $f(x) = \sqrt{x^2 + x}$.

Solution. Since the function is continuous for all x, this function has no vertical asymptotes. In order to find the slant asymptotes, we will first consider $x \to \infty$:

$$k = \lim_{x\to\infty} \frac{\sqrt{x^2+x}}{x} = \lim_{x\to\infty} \sqrt{\frac{x^2+x}{x^2}} = \lim_{x\to\infty} \sqrt{1 + \frac{1}{x}} = 1;$$

$$b = \lim_{x\to\infty} \left(\sqrt{x^2+x} - x \right) = \lim_{x\to\infty} \frac{x}{\sqrt{x^2+x}+x} = \lim_{x\to\infty} \frac{1}{\sqrt{1+\frac{1}{x}}+1} = \frac{1}{2}.$$

Therefore, the asymptote at ∞ is $y = x + \frac{1}{2}$. Consider now $x \to -\infty$: unlike the previous example, here the limits will be different, because for $x < 0$ we have $x = -\sqrt{x^2}$:

$$k = \lim_{x\to-\infty} \frac{\sqrt{x^2+x}}{x} = -\lim_{x\to-\infty} \sqrt{\frac{x^2+x}{x^2}} = -\lim_{x\to-\infty} \sqrt{1 + \frac{1}{x}} = -1;$$

$$b = \lim_{x\to-\infty} \left(\sqrt{x^2+x} + x \right) = \lim_{x\to-\infty} \frac{x}{\sqrt{x^2+x}-x} = \lim_{x\to-\infty} \frac{1}{-\sqrt{1+\frac{1}{x}}-1} = -\frac{1}{2}.$$

The asymptote at $-\infty$ is $y = -x - \frac{1}{2}$.

Find all asymptotes of the following functions.

3.65. $x^2 - y^2 = 1$

3.66. $xy = x + 1$

3.67. $y = \dfrac{1}{x^2+1}$

3.68. $y = \dfrac{x^2+6x-5}{x}$

3.69. $y = \dfrac{x^2}{x+3}$

3.70. $y = \dfrac{x^2-5x+4}{x-4}$

3.71. $y = \dfrac{x^3}{x^2+2x-3}$

3.72. $y = x + e^{-x}$

3.73. $y = \sqrt{x^2-1}$

3.74. $y = \dfrac{x}{\sqrt{x^2+1}}$

3.75. $y = \dfrac{x^2}{\sqrt{x^2-1}}$

3.76. $y = x + \dfrac{\sin x}{x}$

3.77. $y = \sqrt{\frac{1}{x} - 1}$

3.78. $y = e^{1/x}$

3.79. $y = \sqrt{\dfrac{x^3}{x-2}}$

3.80. $y = \sqrt[3]{x^3 - 6x^2}$

3.4 Equivalent infinitely small functions

Definition. The function $f(x)$ is **infinitely small** as $x \to a$, if $\lim\limits_{x\to a} f(x) = 0$.

Definition. Two infinitely small functions are **equivalent** as $x \to a$, if

$$\lim_{x\to a} \frac{f(x)}{g(x)} = 1.$$

Equivalency is denoted $f(x) \sim g(x)$.

Table of equivalent infinitely small functions

1. $\sin x \sim x$
2. $\tan x \sim x$
3. $1 - \cos x \sim \dfrac{x^2}{2}$
4. $\tan^{-1} x \sim x$
5. $\sin^{-1} x \sim x$
6. $\ln(1+x) \sim x$
7. $e^x - 1 \sim x$
8. $(1+x)^n - 1 \sim nx$

Example 3.8. Find the following limits.

a) $\lim\limits_{x\to 0} \dfrac{\sin 2x}{7x}$; b) $\lim\limits_{x\to 0} \dfrac{\tan^{-1} 5x}{\sin^{-1} 9x}$; c) $\lim\limits_{x\to \pi/2} \dfrac{\cos x}{\pi - 2x}$; d) $\lim\limits_{x\to 0} \dfrac{\tan x - \sin x}{x^3}$.

Solution.

a) $\lim\limits_{x\to 0} \dfrac{\sin 2x}{7x} = \lim\limits_{x\to 0} \dfrac{\sin 2x}{2x} \cdot \dfrac{2x}{7x} = 1 \cdot \dfrac{2}{7} = \dfrac{2}{7}$.

b) $\lim\limits_{x\to 0} \dfrac{\tan^{-1} 5x}{\sin^{-1} 9x} = \lim\limits_{x\to 0} \dfrac{\tan^{-1} 5x}{5x} \cdot \dfrac{9x}{\sin^{-1} 9x} \cdot \dfrac{5x}{9x} = 1 \cdot 1 \cdot \dfrac{5}{9} = \dfrac{5}{9}$.

c) $\lim\limits_{x\to \pi/2} \dfrac{\cos x}{\pi - 2x} = \left\{ \begin{array}{l} y = \frac{\pi}{2} - x, \\ y \to 0 \end{array} \right\} = \lim\limits_{y\to 0} \dfrac{\cos\left(\frac{\pi}{2} - y\right)}{2y} = \lim\limits_{y\to 0} \dfrac{\sin y}{2y} = \dfrac{1}{2}$.

d) $\lim\limits_{x\to 0} \dfrac{\tan x - \sin x}{x^3} = \lim\limits_{x\to 0} \dfrac{\sin x}{x} \cdot \dfrac{1 - \cos x}{x^2 \cos x} = 1 \cdot \lim\limits_{x\to 0} \dfrac{1 - \cos x}{x^2} = \dfrac{1}{2}$.

Find the following limits using equivalency.

3.81. $\lim\limits_{x\to 0} \dfrac{\sin 3x}{x}$

3.82. $\lim\limits_{x\to 0} \dfrac{\sin 3x}{\sin 2x}$

3.83. $\lim\limits_{x\to 0} \dfrac{\tan 2x}{\tan(-5x)}$

3.84. $\lim\limits_{x\to 0} \dfrac{\sin 3x}{\tan 7x}$

3.85. $\lim\limits_{x\to 0} \dfrac{1-\cos x}{x^2}$

3.86. $\lim\limits_{x\to 0} \dfrac{\sin^{-1} x}{x}$

3.87. $\lim\limits_{x\to 0} \dfrac{1-\cos 2x}{1-\cos 3x}$

3.88. $\lim\limits_{x\to 0} \dfrac{\sin^{-1} 2x}{\tan^{-1} 3x}$

3.89. $\lim\limits_{x\to 0} \dfrac{\sin 2x - \sin x}{\sin 3x}$

3.90. $\lim\limits_{x\to 0} \dfrac{\sin^2 x - \sin 2x}{x}$

3.91. $\lim\limits_{x\to 0} \dfrac{\cos 2x - \cos x}{\sin^2 x}$

3.92. $\lim\limits_{x\to \pi} \dfrac{\sin x}{\pi^2 - x^2}$

3.93. $\lim\limits_{x\to \pi/2} \dfrac{1-\sin x}{(\pi/2 - x)^2}$

3.94. $\lim\limits_{x\to \pi/4} \dfrac{\sin x - \cos x}{\pi - 4x}$

3.95. $\lim\limits_{x\to 1} \dfrac{\sin \pi x}{\sin 2\pi x}$

3.96. $\lim\limits_{x\to 2} \cot(\pi x)(2-x)$

3.97. $\lim\limits_{x\to \pi} \dfrac{\sin^2 x}{1+\cos x}$

3.98. $\lim\limits_{x\to \pi/4} \dfrac{\sin x - \cos x}{\tan x - 1}$

3.99. $\lim\limits_{x\to \pi/2} \dfrac{1-\sqrt{1+\cot x}}{\cot x}$

3.100. $\lim\limits_{x\to 0} \dfrac{\tan x}{\sqrt{1-\tan x} - 1}$

3.101. $\lim\limits_{x\to \pi/2} \dfrac{2\cos x}{\pi - 2x}$

3.102. $\lim\limits_{x\to \pi/2} \dfrac{\pi - 2x}{2\cot x}$

3.103. $\lim\limits_{x\to 0} \dfrac{\log_a(1+2x)}{x}$

3.104. $\lim\limits_{x\to 0} \dfrac{a^x - 1}{3x}$

3.105. $\lim\limits_{x\to 0} \dfrac{\sin 4x}{e^{-3x} - 1}$

3.106. $\lim\limits_{x\to 0} \dfrac{\ln(1+2x)}{\tan 4x}$

3.107. $\lim\limits_{x\to 0} \dfrac{\sqrt{\frac{x+3}{3-2x}} - 1}{x + \sin x}$

3.108. $\lim\limits_{x\to -1} \dfrac{\cos 4x - \cos 4}{\sin 4 + \sin 4x}$

3.109. $\lim\limits_{x\to 0} \dfrac{1-e^{-x}}{\sin 3x}$

3.110. $\lim\limits_{x\to 0} \dfrac{e^{-3x} - e^{-4x}}{x}$

3.111. $\lim\limits_{x\to 0} \dfrac{3^{2x} - 1}{x}$

3.112. $\lim\limits_{x\to 0} \dfrac{e^{\sin 2x} - e^{\sin x}}{x}$

3.113. $\lim\limits_{\triangle x\to 0} \dfrac{\sin(x+\triangle x) - 2\sin x + \sin(x - \triangle x)}{\triangle x^2}$

3.114. $\lim\limits_{\triangle x\to 0} \dfrac{\cos(x+\triangle x) - 2\cos x + \cos(x - \triangle x)}{\triangle x^2}$

Chapter 4.
CONTINUOUS FUNCTIONS

4.1 Continuity

Definition. The function f is **continuous** at $x=a$ if $\lim\limits_{x\to a} f(x)=f(a)$.

This definition is equivalent to the condition $\lim\limits_{x\to a^-} f(x)=\lim\limits_{x\to a^+} f(x)=f(a)$.

Intuitively, continuous functions have a graph which can be drawn without lifting the pencil.

Example 4.1. Define $f(1)$ so that the function $f(x)=\dfrac{x^2-1}{x^3-1}$ is continuous at $x=1$.

Solution. By the definition, $f(x)$ will be continuous at $x=1$ if $\lim\limits_{x\to 1} f(x)=f(1)$.

$$\lim_{x\to 1} f(x)=\lim_{x\to 1}\frac{x^2-1}{x^3-1}=\lim_{x\to 1}\frac{x+1}{x^2+x+1}=\frac{2}{3}.$$

Therefore, $f(x)$ will be continuous at $x=1$ if $f(1)=\frac{2}{3}$.

Define $f(0)$ so that the following functions will be continuous at $x=0$.

4.1. $f(x)=\dfrac{\sqrt{1+x}-1}{x}$.

4.2. $f(x)=\dfrac{\sqrt{1+2x}-1}{\sqrt[3]{1+2x}-1}$.

4.3. $f(x)=\dfrac{\sin x}{x}$.

4.4. $f(x)=4\dfrac{1-\cos x}{x^2}$.

4.5. $f(x)=\dfrac{\tan 2x}{x}$.

4.6. $f(x)=\sin x\sin\dfrac{1}{x}$.

4.7. $f(x)=(1+x)^{1/x}$ $(x>0)$.

4.8. $f(x)=e^{-1/x^2}$.

4.9. If the function $f(x)$ is continuous for all x and $f(x)=\dfrac{x^2-5x+4}{x-4}$ when $x\neq 4$, what is $f(4)$?

4.10. If possible, define $f(1)$ so that the function $f(x)=\dfrac{\sqrt{x^2-2x+1}}{x^2-4x+3}$ is continuous at $x=1$.

Example 4.2. Determine the value of k such that the function

$$f(x)=\begin{cases}3kx-5, & x<2;\\ 4x-5k, & x\geq 2\end{cases}$$

is continuous.

Solution. Consider the point $x = 2$.

$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} (3kx-5) = 6k-5;$

$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} (4x-5k) = 8-5k;$

$f(2) = 4\times 2 - 5k = 8-5k.$

The function $f(x)$ will be continuous at $x=2$ if $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x) = f(2)$, so for continuity put

$6k-5 = 8-5k \quad \Rightarrow \quad 11k = 13 \quad \Rightarrow \quad k = \frac{13}{11}.$

Find the values of all unknown constants so that the function is continuous.

4.11. $f(x) = \begin{cases} x+1, & x \le 1, \\ 3-mx^2, & x>1. \end{cases}$

4.12. $f(x) = \begin{cases} 4kx-4, & x>2, \\ 4x-2k, & x\le 2. \end{cases}$

4.13. $f(x) = \begin{cases} \dfrac{x^2-9}{x-3}, & x \ne 3, \\ A, & x=3. \end{cases}$

4.14. $f(x) = \begin{cases} e^{2x}, & x<0, \\ x-a, & x\ge 0. \end{cases}$

4.15. $f(x) = \begin{cases} x^2+3x, & x\le 2, \\ bx\ln x, & x>2. \end{cases}$

4.16. $f(x) = \begin{cases} e^{2x+d}, & x\ge 0, \\ x+2, & x<0. \end{cases}$

4.17. $f(x) = \begin{cases} x+2, & x<k, \\ -x+6, & x\ge k. \end{cases}$

4.18. $f(x) = \begin{cases} x^2+ax, & x<2, \\ \sin\dfrac{\pi x}{2} - 2x, & x\ge 2. \end{cases}$

4.19. $f(x) = \begin{cases} (x+c)^2, & x<3, \\ 5x+c, & x\ge 3. \end{cases}$

4.20. $f(x) = \begin{cases} e^{n+x}, & x\ge 0, \\ x+2, & x<0. \end{cases}$

4.21. $f(x) = \begin{cases} 2x, & x\le -1, \\ ax+b, & |x|<1, \\ x^2+3, & x\ge 1. \end{cases}$

4.22. $f(x) = \begin{cases} -2\sin x, & x< -\dfrac{\pi}{2}, \\ a\sin x + b, & |x| \le \dfrac{\pi}{2}, \\ \cos x, & x>\dfrac{\pi}{2}. \end{cases}$

4.23. Is it true that the square of a discontinuous function is also discontinuous?

4.24. Prove that the cube of a discontinuous function is also discontinuous.

4.25. If $f(x)$ is continuous at $x = x_0$, and $g(x)$ is not, what can be said about the continuity of the functions $f(x)+g(x)$ and $f(x)g(x)$?

4.26. If $f(x)$ and $g(x)$ are not continuous at $x = x_0$, what can be said about the continuity of the functions $f(x)+g(x)$ and $f(x)g(x)$?

4.27. Show that the equation $x^5 - 3x = 1$ has at least one root between 1 and 2.

4.28. Show that the equation $(3-x)\cdot 3^x = 3$ has at least one root on the interval $(2,3)$.

4.29. Let $f(x)$ be continuous on $[a,b]$. Prove that for any points x_1 and x_2 in $[a,b]$ there exists a point c such that

$$f(c) = \frac{1}{2}\left(f(x_1)+f(x_2)\right).$$

4.30. Consider the function $y = f(x)$, where $f(x) = 0$ if x is a rational number, and $f(x) = x$ if x is irrational. At how many points is this function continuous?

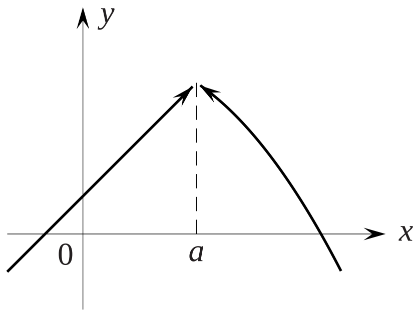
4.2 Points of discontinuity

There are three types of discontinuities:

I. Removable discontinuities

Definition. A discontinuity is classified as **removable**, if $\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x)$.
(An equivalent statement is that $\lim_{x\to a} f(x)$ exists.)

Since f is not continuous at $x = a$, a removable discontinuity implies that $f(a)$ is either undefined, or defined so that $\lim_{x\to a} f(x) \neq f(a)$. By redefining $f(a)$, it is possible to make f continuous at $x = a$.

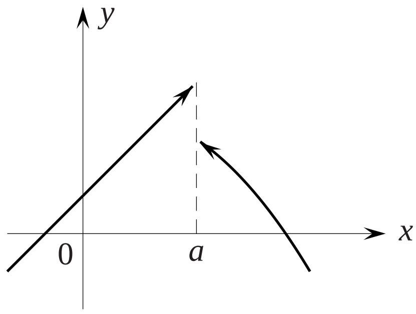


Definition. All discontinuities which are not removable are called **unremovable**.

II. Jump discontinuities

Definition. The function f has a **jump discontinuity** at $x = a$, if both the left and right limits of f exist at that point, but $\lim_{x\to a^-} f(x) \neq \lim_{x\to a^+} f(x)$.

The genesis of the term "jump discontinuity" should be obvious just by looking at the graph to the right...

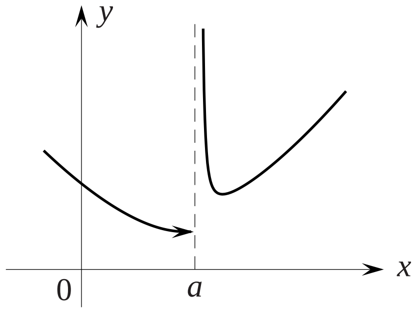


Removable and jump discontinuities are also sometimes called **Type I discontinuities**. Using this terminology, a jump discontinuity would be called a "non-removable Type I discontinuity".

III. Type II discontinuities

Definition. The function f has a **Type II discontinuity** at $x = a$, if either one or both of the one-sided limits of f do not exist at that point.

In particular, if one of the one-sided limits is infinite, then f has a Type II discontinuity at that point. *However, this is only a special case of a Type II discontinuity.*



Example 4.3. Classify the points of discontinuity of the function

$$f(x) = \begin{cases} x^2, & -2 \le x < 0; \\ 4, & x = 0; \\ \dfrac{1}{x}, & 0 < x \le 2. \end{cases}$$

Solution. The function $f(x)$ is defined on the interval $[-2,2]$. Since the function x^2 is continuous on the interval $[-2,0)$ and the function $1/x$ is continuous on $(0,2]$, the only point that needs to be considered is $x = 0$.

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} x^2 = 0; \quad \lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{1}{x} = \infty.$$

Since the right limit does not exist (equals infinity), $f(x)$ has a type II discontinuity at $x = 0$.

Example 4.4. Classify the points of discontinuity of the function

$$f(x) = \begin{cases} 0, & x < 0; \\ x, & 0 \le x < 1; \\ -x^2 + 4x + 2, & 1 \le x < 3; \\ 4 - x, & x \ge 3. \end{cases}$$

Solution. The points that are possible points of discontinuity are $x = 0$, $x = 1$, and $x = 3$.

$\lim\limits_{x\to 0^-} f(x) = \lim\limits_{x\to 0^-} 0 = 0$; $\lim\limits_{x\to 0^+} f(x) = \lim\limits_{x\to 0^+} x = 0$. Since $f(0) = 0$ as well, $f(x)$ is continuous at $x = 0$.

$\lim\limits_{x\to 1^-} f(x) = \lim\limits_{x\to 1^-} x = 1$; $\lim\limits_{x\to 1^+} f(x) = \lim\limits_{x\to 1^+} (-x^2 + 4x + 2) = 5$. Since the left and right limits of $f(x)$ exist but are not equal to each other, $f(x)$ has a type I discontinuity at $x = 1$.

$\lim\limits_{x\to 3^-} f(x) = \lim\limits_{x\to 3^-} (-x^2 + 4x + 2) = 5$; $\lim\limits_{x\to 3^+} f(x) = \lim\limits_{x\to 3^+} (4 - x) = 1$. Since the left and right limits of $f(x)$ exist but are not equal to each other, $f(x)$ has a type I discontinuity at $x = 3$.

Find the points of discontinuity and classify them.

4.31. $f(x)=\dfrac{x}{|x|}$

4.32. $f(x)=\dfrac{3}{x+4}$

4.33. $f(x)=\dfrac{1}{(x+4)^2}$

4.34. $f(x)=\dfrac{x^2+x-6}{|x-2|}$

4.35. $f(x)=\begin{cases}4+x, & x\le 1,\\ 2-x, & x>1\end{cases}$

4.36. $f(x)=\dfrac{x^2-4}{x^2-5x+6}$

4.37. $f(x)=\dfrac{x^2-1}{x^3-3x+2}$

4.38. $f(x)=\dfrac{|x-2|}{x^2-4}$

4.39. $f(x)=\dfrac{5}{2^{1/x}-2}$

4.40. $f(x)=2^{2^{1/(1-x)}}$

4.41. $f(x)=\dfrac{5^{1/x}-1}{5^{1/x}+1}$

4.42. $f(x)=\begin{cases}\dfrac{\sin x}{x}, & x\ne 0,\\ 0, & x=0\end{cases}$

4.43. $f(x)=\dfrac{\sin x}{x}$

4.44. $f(x)=\dfrac{\cos x}{x}$

4.45. $f(x)=\begin{cases}\cos\dfrac{\pi x}{2}, & |x|\le 1,\\ |x-1|, & |x|>1\end{cases}$

4.46. $f(x)=\dfrac{\sqrt{1-\cos x}}{x}$

4.47. $f(x)=\tan^{-1}\dfrac{1}{x}$

4.48. $f(x)=\dfrac{1}{\ln|x|}$

4.49. $f(x)=\begin{cases}\cot^2\pi x, & x\notin\mathbb{Z},\\ 0, & x\in\mathbb{Z}\end{cases}$

4.50. $f(x)=\dfrac{1}{1+3^{\tan x}}$

4.51. $f(x)=\tan^{-1}\left(\dfrac{1}{x}+\dfrac{1}{x-1}+\dfrac{1}{x-2}\right).$

4.3 Properties of continuous functions

Theorem (Boundedness theorem)

If f is continuous on the *closed* interval $[a,b]$, then it is bounded on $[a,b]$, i.e. there exist m and M such that $m\le f(x)\le M$ for all $x\in[a,b]$.

Theorem (Extreme Value theorem)

If f is continuous on the *closed* interval $[a,b]$, then f attains its *minimum* and *maximum* values on $[a,b]$, i.e. there exists $c_1\in[a,b]$ such that $f(x)\ge f(c_1)$, $x\in[a,b]$, and there exists $c_2\in[a,b]$ such that $f(x)\le f(c_2)$, $x\in[a,b]$.

Theorem (Intermediate Value theorem)

If f is continuous on the *closed* interval $[a,b]$, m is its minimum value and M is its maximum value on $[a,b]$, then for any μ, $m<\mu<M$, there exists $c\in[a,b]$ such that $f(c)=\mu$.

A special case of the intermediate value theorem is the

Theorem (Root theorem)

If f is continuous on the *closed* interval $[a,b]$ and its values at the end-points of the interval have different signs (i.e. $f(a)f(b) < 0$), then there exists $c \in (a,b)$ such that $f(c) = 0$.

4.52. Prove that if $f(x)$ is continuous on (a,b) and x_1, x_2 and x_3 belong to (a,b), then there exists $c \in (a,b)$ such that $f(c) = \frac{1}{3}(f(x_1) + f(x_2) + f(x_3))$.

4.53. Prove that any polynomial with an odd highest power has at least one root.

4.54. Prove that if f and g are continuous on $[a,b]$ and $f(a) > g(a)$, $f(b) < g(b)$, then there exists a point $c \in [a,b]$ such that $f(c) = g(c)$.

4.55. Does there exist a function which is continuous on $[a,b]$ and the range of which is $[0,1] \cup [2,3]$?

Chapter 5.
DERIVATIVES

5.1 Definition of the derivative

Definition. Given two points x and $x_1 = x + \triangle x$, the **increment** of $f(x)$ on the interval $[x, x_1]$ is given by

$$\triangle f = f(x_1) - f(x) = f(x + \triangle x) - f(x).$$

Note that $\triangle x$ (called the **increment** of x) does not always have to be positive!

Definition. The **derivative** of the function $f(x)$ is the limit

$$f'(x) = \lim_{\triangle x \to 0} \frac{\triangle f}{\triangle x} = \lim_{\triangle x \to 0} \frac{f(x + \triangle x) - f(x)}{\triangle x}.$$

The derivative can also be denoted by $\dfrac{df}{dx}$.

The derivative at $x = x_0$ can be expressed in other ways as well:

$$f'(x_0) = \lim_{\triangle x \to 0} \frac{f(x_0 + \triangle x) - f(x_0)}{\triangle x} =$$

$$= \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}.$$

Definition. The **second derivative** of the function $f(x)$ is the derivative of $f'(x)$:

$$f''(x) = \frac{d^2 f}{dx^2} = \lim_{\triangle x \to 0} \frac{f'(x + \triangle x) - f'(x)}{\triangle x}.$$

Definition. A function is **differentiable**, if the main part of its increment is linear with respect to the increment of the argument, i.e., if

$$\triangle y = A \triangle x + \alpha(\triangle x) \triangle x$$

where $\lim\limits_{\triangle x \to 0} \alpha(\triangle x) = 0$.

Theorem. A function is differentiable if and only if it has a derivative.

A function that can be differentiated twice (i.e., a function that has a second derivative) is called **twice-differentiable**.

Approximate calculation of the derivative

If it is not possible to find the exact expression for the derivative (for instance, if $f(x)$ is defined by a graph or by a table), then the approximate value of $f'(x)$ at the point $x=x_0$ equals

$$f'(x_0)\approx\frac{\triangle f}{\triangle x}=\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x},$$

where $\triangle x=x-x_0$. Remember that $\triangle x$ does not always have to be positive. In practice, you can use any of the following formulas:

$$f'(x_0)\approx\frac{f(x_0+h)-f(x_0)}{h};$$
$$f'(x_0)\approx\frac{f(x_0)-f(x_0-h)}{h};$$
$$f'(x_0)\approx\frac{f(x_0+h)-f(x_0-h)}{2h}.$$

The second derivative can be approximated by using the formula

$$f''(x_0)\approx\frac{f(x_0+h)-2f(x_0)+f(x_0-h)}{h^2}.$$

Example 5.1. Find the derivative of $f(x)=4x^2$ using the definition.

Solution.

$$f(x+\triangle x)-f(x)=4(x+\triangle x)^2-4x^2=4x^2+8x\triangle x+4(\triangle x)^2-4x^2=$$
$$=8x\triangle x+4(\triangle x)^2.$$

Therefore,

$$f'(x)=\lim_{\triangle x\to 0}\frac{f(x+\triangle x)-f(x)}{\triangle x}=\lim_{\triangle x\to 0}\frac{8x\triangle x+4(\triangle x)^2}{\triangle x}=\lim_{\triangle x\to 0}(8x+4\triangle x)=8x.$$

Find the derivatives of the following functions using the definition.

5.1. x

5.2. $\frac{1}{x}$

5.3. x^3

5.4. x^2+2x+2

5.5. $\sqrt{x}$

5.6. $4\sin\frac{x}{2}$

5.7. $\frac{3}{\sqrt{x}}+2\cdot 3^x$

5.8. $\log_a x$

5.9. $\log_2(3x)$

5.10. $3\sqrt{x}+\log_2\frac{x}{2}$

Example 5.2. Find an approximate value of $f'(2)$ and $f''(2)$ if it is known that $f(1.8) = 3.3$, $f(2) = 3.5$ and $f(2.2) = 3.6$.

Solution. There are three equally valid approximate values of $f'(2)$ that can be found:

$$f'(2) \approx \frac{f(2.2) - f(2)}{2.2 - 2} = \frac{3.6 - 3.5}{.2} = \frac{1}{2};$$

$$f'(2) \approx \frac{f(2) - f(1.8)}{2 - 1.8} = \frac{3.5 - 3.3}{.2} = 1;$$

$$f'(2) \approx \frac{f(2.2) - f(1.8)}{2.2 - 1.8} = \frac{3.6 - 3.3}{.4} = \frac{3}{4}.$$

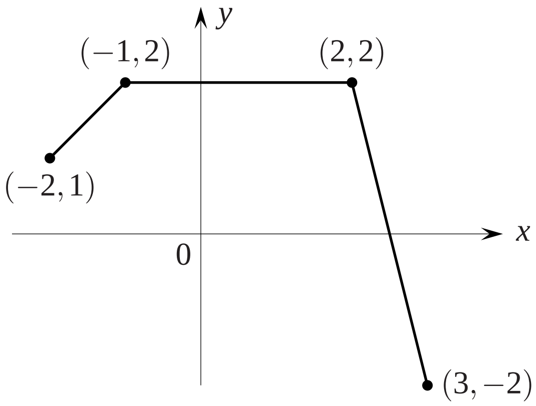
Remember that we have no reason to consider any of these approximate values to be more accurate than the rest.

Now find an approximation for $f''(2)$:

$$f''(2) \approx \frac{f(2.2) - 2f(2) + f(1.8)}{.2^2} = -2.5.$$

5.11. Use the table to find an approximate value of $f'(3)$ and $f''(3)$.

x	2.95	3	3.05
$f(x)$	1.07	1.12	1.18



5.12. The graph of $f(x)$, which consists of three line segments, is shown above. Find $f'(-1.5)$, $f'(1)$, and $f'(2.5)$.

5.2 Differentiation of explicit functions

Rules of differentiation

1. $\big(kf(x)\big)' = kf'(x)$, where k is a constant;
2. (the sum rule) $\big(f(x) + g(x)\big)' = f'(x) + g'(x)$;

3. (the product rule) $(f(x)g(x))' = f'(x)g(x) + f(x)g'(x)$;

4. (the quotient rule) $\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}$;

4. (the chain rule) $\big(f(g(x))\big)' = f'_g(g(x))g'_x(x)$.

Table of derivatives

1. $(x^n)' = nx^{n-1}$
2. $(a^x)' = a^x \ln a, \quad (e^x)' = e^x$
3. $(\log_a x)' = \frac{1}{x\ln a}, \quad (\ln x)' = \frac{1}{x}$
4. $(\sin x)' = \cos x$
5. $(\cos x)' = -\sin x$
6. $(\tan x)' = \frac{1}{\cos^2 x}$
7. $(\cot x)' = -\frac{1}{\sin^2 x}$
8. $\left(\sin^{-1} x\right)' = \frac{1}{\sqrt{1-x^2}}$
9. $\left(\cos^{-1} x\right)' = -\frac{1}{\sqrt{1-x^2}}$
10. $\left(\tan^{-1} x\right)' = \frac{1}{1+x^2}$

11 $\left(\cot^{-1} x\right)' = -\frac{1}{1+x^2}$

Example 5.3. Find the derivatives of the following functions:

$$\text{a) } y = x^3 \tan^{-1} x; \quad \text{b) } y = \frac{x^2+x-1}{x^3+1}; \quad \text{c) } y = \sqrt{1+2\tan x}.$$

Solution. a) Using the product rule,

$$\left(x^3 \tan^{-1} x\right)' = (x^3)' \tan^{-1} x + x^3 \left(\tan^{-1} x\right)' = 3x^2 \tan^{-1} x + \frac{x^3}{1+x^2}.$$

b) Using the quotient rule,

$$\left(\frac{x^2+x-1}{x^3+1}\right)' = \frac{(x^2+x-1)'(x^3+1) - (x^2+x-1)(x^3+1)'}{(x^3+1)^2} =$$

$$= \frac{(2x+1)(x^3+1) - (x^2+x-1)3x^2}{(x^3+1)^2} = \frac{-x^4 - 2x^3 + 3x^2 + 2x + 1}{(x^3+1)^2}.$$

c) Here it is necessary to use the chain rule. The “outer” function is $\sqrt{x}$; the “inner” function is $1+2\tan x$. Their derivatives are, respectively, $1/(2\sqrt{x})$ and $2/\cos^2 x$. Therefore, bearing in mind that the argument of the outer function is $1+2\tan x$,

$$\left(\sqrt{1+2\tan x}\right)' = \frac{1}{2\sqrt{1+2\tan x}} \cdot \frac{2}{\cos^2 x} = \frac{1}{\cos^2 x\sqrt{1+2\tan x}}.$$

Example 5.4. Find the derivative of x^x.

Solution. This function cannot be differentiated in this form. It is first necessary to rewrite it, in order to be able to use the table of elementary derivatives and the rules of differentiation:

$$x^x = e^{\ln(x^x)} = e^{x\ln x}$$

Therefore,

$$(x^x)' = \left(e^{x\ln x}\right)' = e^{x\ln x}(x\ln x)' = e^{x\ln x}(\ln x+1) = x^x(\ln x+1).$$

Find the derivatives of the following functions.

5.13. $5x^2+4x+2$

5.14. $(x^2+4x-12)^{10}$

5.15. $\dfrac{x+2}{x+3}$

5.16. $\dfrac{1}{(x^2-4)^4}$

5.17. $\dfrac{x^2+3x+2}{2x^2+4x+3}$

5.18. $\dfrac{x^2+2x}{x^2+1}$

5.19. $\dfrac{(x^4+1)^3}{(x^3+1)^2}$

5.20. $\dfrac{x^3+2x+1}{x^3+2x^2+1}$

5.21. $\sqrt{4-x^2}$

5.22. $(x^2+6)\sqrt{x^2-3}$

5.23. $\sqrt[3]{(x^3+1)^2}$

5.24. $\dfrac{3x+2}{\sqrt{2-3x}}$

5.25. $\dfrac{x}{\sqrt[3]{x^3+1}}$

5.26. $\dfrac{x^2+x}{\sqrt[3]{x^2-x}}$

5.27. $\dfrac{\sqrt[4]{x}+1}{\sqrt[3]{x}+1}$

5.28. $\sqrt{\dfrac{x+1}{x-1}}$

5.29. $\sqrt{\dfrac{1}{\sqrt[3]{x}+1}}$

5.30. $\dfrac{1}{\sqrt{x+1}+\sqrt{x-1}}$

5.31. $-\dfrac{1}{25}x^3(2-5x^3)^{5/3} - \dfrac{3}{1000}(2-5x^3)^{8/3}$

5.32. $\dfrac{2-\cos x}{2+\cos x}$

5.33. $\dfrac{\tan x-1}{\tan x}$

5.34. $\cos(4x^2+3x+4)$

5.35. $\sqrt{\cos x}$

5.36. $\dfrac{1}{\sqrt{\cos x}}$

5.37. $\dfrac{\sin^2 x}{1+\sin^2 x}$

5.38. $\tan(x^3)$

5.39. $\sqrt{\cot x}$

5.40. $\sin(x^2+\tan x)$

5.41. $\tan(\sin x)$

5.42. $\dfrac{2}{9}\sqrt{\tan^9 x}+\dfrac{2}{5}\sqrt{\tan^5 x}$

5.43. $x-\tan x+\dfrac{1}{3}\tan^3 x$

5.44. $2\tan\dfrac{x}{2}-x+2$

5.45. $2\sin^{-1}x+3\cos^{-1}x$

5.46. $\dfrac{1}{2\sqrt{3}}\tan^{-1}\left(\dfrac{2x^4+1}{\sqrt{3}}\right)$

5.47. $\tan^{-1}\sin x$

5.48. $\sin^{-1}\sqrt{x-1}$

5.49. $\sin^{-1}\dfrac{x-1}{x+1}$

5.50. $\tan^{-1}\dfrac{x^2+1}{x^2-1}$

5.51. $\tan^{-1}x+\cot^{-1}x$

5.52. $\tan^{-1}x^2$

5.53. $\sin^{-1}x-x\sqrt{1-x^2}$

5.54. $\tan^{-1}x-\dfrac{x}{1+x^2}$

5.55. $\dfrac{1}{2}\tan^{-1}(\sin^2 x)$

5.56. $\dfrac{(x^2+1)^2}{4}\cot^{-1}x+\dfrac{x^3+3x}{12}$

5.57. $(\sin^{-1}x)^2+2x\sqrt{1-x^2}\sin^{-1}x-x^2$

5.58. $\ln(2x^2+x+1)$

5.59. $\ln\dfrac{x^2+1}{x^2-1}$

5.60. $\ln\sqrt{x+1}$

5.61. $\ln^3(x^2+1)$

5.62. $\ln\sin x$

5.63. $\dfrac{2}{\ln 2x}$

5.64. $\ln\left(x-\sqrt{x^2+1}\right)$

5.65. $\ln\sqrt{\dfrac{1+x}{1-x}}$

5.66. $\ln\dfrac{x}{1+\sqrt{1+x^2}}$

5.67. $\ln\dfrac{\sqrt{x}-1}{\sqrt{x}+1}$

5.68. $\ln(x+\ln x)$

5.69. $\dfrac{x^2}{2}\ln(x^4+4)+2\tan^{-1}\dfrac{x^2}{2}-x^2$

5.70. xe^{2x}

5.71. $\dfrac{3^x}{3^x+1}$

5.72. $\dfrac{e^x}{x+1}$

5.73. $-\dfrac{1}{2}\ln^2\dfrac{x+1}{x}$

5.74. $\dfrac{x^2-1}{2}\ln\dfrac{1+x}{1-x}+x$

5.75. $\dfrac{\cos x}{2}-\dfrac{3\sqrt{2}}{8}\ln\dfrac{\sqrt{2}\cos x-1}{\sqrt{2}\cos x+1}$

5.76. $\ln\left(\tan\dfrac{x}{2}\right)-\cos x\ln\tan x$

5.77. $e^x(\sin x+\cos x)$

5.78. $(x+1)^{x^2+1}$

5.79. $(\sin x)^x$

5.80. $(\sin x)^{\sin x}$

5.81. $(\tan x)^{\ln x}$

5.82. $|x^3|$

5.83. $|\sin^3 x|$

5.84. If $f(x)$ is differentiable at $x=x_0$, and $g(x)$ is not, what can be said about the differentiability of the functions $f(x)+g(x)$ and $f(x)g(x)$?

5.85. If $f(x)$ and $g(x)$ are not differentiable at $x=x_0$, what can be said about the differentiability of the functions $f(x)+g(x)$ and $f(x)g(x)$?

5.86. Prove that the derivative of a periodic function is also periodic and has the same period.

5.87. Prove that the derivative of an even function is odd, and that the derivative of an odd function is even.

5.88. Prove that the function

$$f(x)=\begin{cases} x^2\sin\dfrac{1}{x}, & x\neq 0;\\ 0, & x=0\end{cases}$$

is differentiable for all x, but its derivative is not continuous.

5.89. Show that if a differentiable function has a discontinuous derivative, then that discontinuity is of type II. (See the previous problem.)

5.90. Show that the function

$$f(x)=\begin{cases} x^2, & x\geq 0;\\ -x^2, & x<0\end{cases}$$

is differentiable at $x=0$, but it is not twice differentiable at $x=0$.

5.91. Prove that the function $y=-\ln(x+1)$ satisfies the equation

$$xy'+1=e^y.$$

5.92. Prove that the function $y=\dfrac{\sin^{-1}x}{\sqrt{1-x^2}}$ satisfies the equation

$$(1-x^2)y'-xy=1.$$

Define the constants a and b so that the following functions will be differentiable.

5.93. $f(x)=\begin{cases} x^2, & x\leq 2;\\ ax+b, & x>2.\end{cases}$

5.94. $f(x)=\begin{cases} x^2, & x\leq x_0;\\ ax+b, & x>x_0.\end{cases}$

5.95. $F(x)=\begin{cases} f(x), & x\leq x_0;\\ ax+b, & x>x_0,\end{cases}$ where $f(x)$ is differentiable at x_0.

Find the following sums:

5.96. $1+2x+3x^2+\ldots+nx^{n-1}$.

5.97. $2+2\cdot 3x+3\cdot 4x^2+\ldots+(n-1)nx^{n-2}$.

5.98. Compare the derivatives of $y=\tan^{-1}\frac{1+x}{1-x}$ and $y=\tan^{-1}x$. What is the relationship between these two functions?

5.99. Prove that $2\tan^{-1}x+\sin^{-1}\frac{2x}{1+x^2}=\pi\,\mathrm{sign}\,x$ for $|x|\geq 1$.

5.3 Differentiation of inverse functions

Definition. The **inverse function** of $y=f(x)$ is the function $x=f^{-1}(y)$; in other words,

$$f^{-1}(f(x))=x, \quad \text{or} \quad f((f^{-1}(x))=x.$$

Theorem. Suppose that f is a function which is differentiable on an open interval containing x_0. If either $f'(x) > 0$ or $f'(x) < 0$ for all x in this interval, then f has a differentiable inverse f^{-1} at $y_0 = f(x_0)$, and

$$\left.\frac{df^{-1}}{dy}\right|_{y=y_0} = \frac{1}{\left.\frac{df}{dx}\right|_{x=x_0}}, \quad \text{or} \quad x'(y_0) = \frac{1}{y'(x_0)}.$$

The condition that $f'(x) > 0$ or $f'(x) < 0$ for all x in an interval is actually a way of making sure that $f(x)$ has two required properties: it is differentiable and one-to-one, i.e., for every value of y there can only be one value of x.

Example 5.5. Find the derivative of a) $y = \log_a x$ and b) $y = \sin^{-1} x$ using the formula for the derivative of the inverse function.

Solution. a) The inverse function for $y = \log_a x$ is $x = a^y$, and its derivative is $x'(y) = a^y \ln a$. Therefore, we have

$$y'(x) = \frac{1}{x'(y)} = \frac{1}{a^y \ln a} = \frac{1}{x \ln a}.$$

b) The inverse function for $y = \sin^{-1} x$ is $x = \sin y$, and its derivative is $x'(y) = \cos y$. We should note, however, that it is necessary to restrict the domain of the function $x = \sin y$ so that we have a one-to-one function; the standard choice is $y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

According to the theorem,

$$\left(\sin^{-1} x\right)' = \frac{1}{(\sin y)'} = \frac{1}{\cos y} = \frac{1}{\sqrt{1-\sin^2 y}} = \frac{1}{\sqrt{1-x^2}}.$$

Note that due to the choice of y we have $\cos y \geq 0$, and it was for this reason that we put $\cos y = \sqrt{1-\sin^2 y}$.

Example 5.6. Find the derivative of $f^{-1}(x)$ at $x = 22$, if $f(x) = x^4 + x^3 - x$ and $f^{-1}(22) = 2$.

Solution. Denote $x = 2$, $y = 22$ (so that $22 = 2^4 + 2^3 - 2$). According to the formula,

$$\left.\frac{df^{-1}}{dy}\right|_{y=22} = \frac{1}{\left.\frac{df}{dx}\right|_{x=2}} = \left.\frac{1}{4x^3 + 3x^2 - 1}\right|_{x=2} = \frac{1}{43}.$$

Find the derivative of $f^{-1}(x)$ at $x = 3$.

5.100. $f(x) = 2x^3 + 1$

5.101. $f(x) = x + \cos x + 2$

5.102. $f(x) = x + \sin x + 3$

5.103. $f(x) = 4 - \log_2 x$

5.4 Implicit differentiation

Definition. An **implicit function** is defined by the equation $F(x,y)=0$.

The derivative of an implicit function can be found using the equality

$$\frac{d}{dx}[F(x,y(x))]=0.$$

Example 5.7. Find the derivative of the functions

$$\text{a) } x^2+y^2-a^2=0; \quad \text{b) } y^6-y-x^2=0; \quad \text{c) } y-x-\frac{1}{4}\sin y=0.$$

Solution. a) Differentiating by x,

$$2x+2yy'=0 \quad \Rightarrow \quad y'=-\frac{x}{y}.$$

b) Differentiating by x,

$$6y^5y'-y'-2x=0 \quad \Rightarrow \quad y'=\frac{2x}{6y^5-1}.$$

c) Differentiating by x,

$$y'-1-\frac{1}{4}(\cos y)y'=0 \quad \Rightarrow \quad y'=\frac{4}{4-\cos y}.$$

Find the derivative of the following implicit functions.

5.104. $x^2+2xy-y^2=2x$

5.105. $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$

5.106. $x^3+y^3-3axy=0$

5.107. $y^3-3y+2ax=0$

5.108. $\cos(xy)=x$

5.109. $x^{2/3}+y^{2/3}=a^{2/3}$

5.110. $y=x+\tan^{-1}y$

5.111. $\tan^{-1}\left(\dfrac{y}{x}\right)=\ln\sqrt{x^2+y^2}$

5.112. Find $y'(0)$, if y is given by the following system:

$$y(x)=\begin{cases} x^2\cos\dfrac{4}{3x}+\dfrac{x}{2}, & x\neq 0; \\ 0, & x=0. \end{cases}$$

5.113. Prove that the function $y(x)$ defined by the equation $xy-\ln y=1$ also satisfies the equation

$$y^2+(xy-1)\frac{dy}{dx}=0.$$

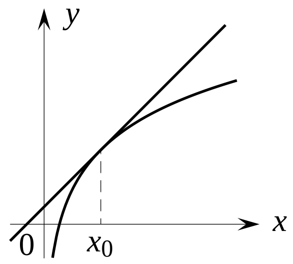
5.114. Prove that if $1+xy=k(x-y)$, where k is some constant, then

$$\frac{dx}{1+x^2}=\frac{dy}{1+y^2}.$$

5.5 Tangent and normal lines

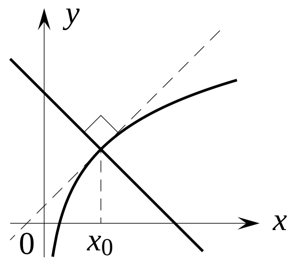
The tangent line:

$y = f'(x_0)(x - x_0) + f(x_0)$

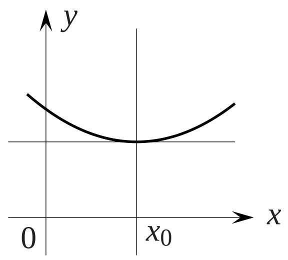


The normal line:

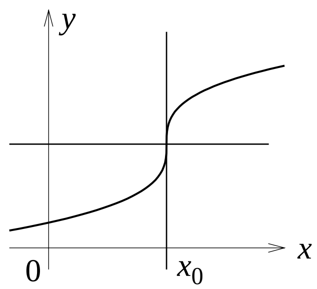
$$y = -\frac{1}{f'(x_0)}(x - x_0) + f(x_0)$$



If $f'(x_0) = 0$, then the graph of $f(x)$ has a **horizontal** tangent line and a **vertical** normal line at x_0.



If $\lim\limits_{x \to x_0} f'(x) = \infty$, then the graph of $f(x)$ has a **vertical** tangent line and a **horizontal** normal line at x_0. Note that in this case $f(x)$ is *not* differentiable at $x = x_0$.



Definition. The function $f(x)$ is **smooth** at $x = x_0$ if the graph of $f(x)$ has a unique tangent line.

Note that a differentiable function is always smooth, but a smooth function can have a vertical tangent line—and therefore it is not differentiable at that point.

Example 5.8. Find the equations of the tangent line and of the normal line drawn to the graph of $y = x^3$ at the point $M(1, 1)$.

Solution. $y'(x) = 3x^2$, so $y'(1) = 3$.

The tangent line: $y = 3(x - 1) + 1 = 3x - 2$.

The normal line: $y = -\frac{1}{3}(x - 1) + 1 = -\frac{1}{3}x + \frac{4}{3}$.

Example 5.9. Find the equation of the tangent line drawn to the graph of $y=x^2$ that is a) parallel to the line $y=4x-5$; b) perpendicular to the line $2x-6y+5=0$.

Solution. a) Parallel lines have equal slopes; the slope of the tangent line at x_0 is given by $y'(x_0)=2x_0$. Therefore,

$$2x_0=4 \quad \Rightarrow \quad x_0=2.$$

The equation of the tangent line at $x_0=2$ is $y=4(x-2)+4$, or $y=4x-4$.

b) The product of the slopes of perpendicular lines is -1. Therefore,

$$2x_0\cdot\frac{1}{3}=-1 \quad \Rightarrow \quad x_0=-\frac{3}{2}.$$

The equation of the tangent line is $y=-3\left(x+\dfrac{3}{2}\right)+\dfrac{9}{4}$, or $y=-3x-\dfrac{9}{4}$.

5.115. Find the points on the curve $y=x^2(x-2)^2$ where the tangent line is parallel to the x-axis.

5.116. Find the equations of the tangent lines drawn to the graph of $y=x-\frac{1}{x}$ at the x-intercepts.

5.117. Find the equation of the tangent line drawn to the graph of $y=\sin x$ at the point $\left(\frac{3\pi}{4},\frac{\sqrt{2}}{2}\right)$.

5.118. Find the equation of the tangent line drawn to the graph of $y=\sin x$ at the point (x_0,y_0).

5.119. Find the equation of the tangent line drawn to the graph of $y=\frac{8a^3}{4a^2+x^2}$ at the point where $x=2a$.

5.120. Find all points on the curve $y=3x^2$ where the tangent line passes through the point $(2,9)$.

5.121. A chord connects two points on the parabola $y=x^2-2x+5$ with $x-$coordinates $x_1=1$ and $x_2=3$. Find the equation of the tangent line drawn to the parabola that is parallel to the chord.

5.122. Show that the tangent lines drawn to the graph of $y=\frac{x-4}{x-2}$ at the x-intercept and y-intercept are parallel.

5.123. Find the equations of the tangent lines drawn to the graph of $y=\frac{x+9}{x+5}$ so that they go through the origin.

5.124. For what value of a will the curves $y=ax^2$ and $y=\ln x$ be tangent to each other?

5.125. At what angle does the graph of $y=\ln x$ intersect the x-axis?

5.126. Find the equation of the tangent line drawn to the graph of $x^2=4ay$ at the point (x_0,y_0) if $x_0=2am$.

5.127. Find the equation of the normal line drawn to the graph of $y=\frac{x^2-3x+6}{x^2}$ at the point where $x=3$.

5.128. Find the points at which the tangent line drawn to the curve $y=x^3+x-2$ is parallel to the line $y=4x-1$.

5.129. Find the normal line drawn to the graph of $y=x\ln x$ which is parallel to the line $2x-2y+3=0$.

5.130. Find the equation of the tangent line drawn to the graph of $y=x^3+3x^2-5$ which is perpendicular to the line $2x-6y+1=0$.

5.131. Find the equations of the tangent line and normal line drawn to the graph of the function $(x+1)\sqrt[3]{3-x}$ at the points a) (-1,0); b) (2,3); c) (3,0).

5.132. Find the equations of the tangent line and normal line drawn to the curve $x^2+2xy^2+3y^4=6$ at the point $M(1,-1)$.

5.133. Prove that the parabola

$$y=a(x-x_1)(x-x_2),$$

where $a\neq 0$, $x_1\neq x_2$, intersects the x-axis at acute angles α and β, where $\alpha=\beta$.

5.134. Find the angle between the left and right tangent lines drawn to the graph of the function

$$f(x)=\sin^{-1}\left(\frac{2x}{1+x^2}\right)$$

at the point $x=1$.

5.6 The differential

Definition. The **differential** of $f(x)$ is given by $df=f'(x)dx$.

If x is sufficiently close to x_0 and $f(x_0)$ is known, then differentials can be used to find an approximate value of $f(x)$:

$$f(x)\approx f(x_0)+df=f(x_0)+f'(x_0)(x-x_0).$$

Note that this can also be understood as the **tangent line approximation** of $f(x)$ at $x=x_0$.

Example 5.10. Using differentials, find the approximate value of 20.1^2.

Solution. Consider the function $f(x)=x^2$. The differential of this function is $df=2xdx$. At $x_0=20$, $f(x_0)=400$ and $f'(x_0)=40$. Therefore,

$$20.1^2\approx 400+40\cdot 0.1=404.$$

Using differentials, find the approximate value of the following expressions. Compare your results to the exact values.

5.135. $\sqrt{120}$

5.136. $\sqrt[4]{80}$

5.137. $\log 11$

5.138. $\sin\left(\frac{\pi}{6}+\frac{\pi}{100}\right)$

5.139. $\cos(151^\circ)$

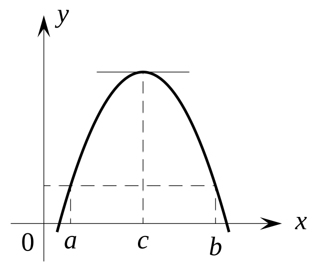
5.140. $\sqrt{\frac{2.027^2-3}{2.027^2+5}}$

5.7 Rolle's Theorem and the Mean Value Theorem

Theorem (Rolle's Theorem)

If $f(x)$ is continuous on $[a,b]$, differentiable on (a,b), and $f(a)=f(b)$, then there exists $c\in(a,b)$ such that $f'(c)=0$.

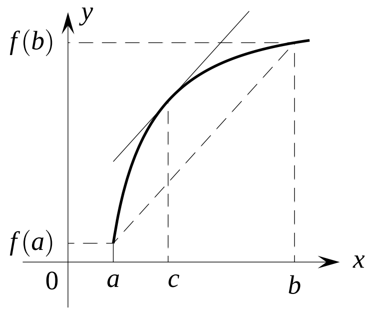
The geometrical interpretation of Rolle's Theorem is that if a function is differentiable and assumes the same value at the ends of an interval, then there is a point where the tangent line drawn to the graph of $f(x)$ is horizontal.



Theorem (The Mean Value Theorem)

If $f(x)$ is continuous on $[a,b]$, differentiable on (a,b), then there exists $c\in(a,b)$ such that $f'(c)=\dfrac{f(b)-f(a)}{b-a}$.

The geometrical interpretation of the Mean Value Theorem can be given as follows. If a secant line is drawn between any two points on the graph of a differentiable function, there exists a point on the graph between these two points at which the tangent line to the graph of f is parallel to the secant.



5.141. Check the validity of Rolle's Theorem for the function

$$f(x)=(x-1)(x-2)(x-3).$$

5.142. Check the validity of Rolle's Theorem for the function

$$f(x)=\sqrt[3]{x^2-3x+2}$$

on the interval $[1,2]$.

5.143. Without finding the derivative of the function

$$f(x)=(x-1)(x-2)(x-3)(x-4),$$

determine how many roots the equation $f'(x)=0$ has and find the intervals where they are located.

5.144. The function $f(x)=\frac{1-x^2}{x^4}$ equals zero at $x=-1$ and $x=1$, and yet $f'(x)\neq 0$ for all values of x, $-1<x<1$. Explain this seeming contradiction to Rolle's Theorem.

5.145. The function $f(x)=1-\sqrt[3]{x^2}$ equals zero at $x=-1$ and $x=1$, and yet $f'(x)\neq 0$ for all values of x, $-1<x<1$. Explain this seeming contradiction to Rolle's Theorem.

5.146. Write the Mean value Theorem for the function $f(x)=x(1-\ln x)$ on the interval $[a,b]$ $(a>0)$.

Find the number in the given interval that satisfies the conclusion of the Mean Value Theorem.

5.147. $f(x) = x^2 - 5x + 7$, $x \in [-1,3]$

5.148. $f(x) = x^3 - 6x^2 + 9x + 2$, $x \in [0,4]$

5.149. $f(x) = x^4 - 16x^2 + 2$, $x \in [-1,3]$

5.150. Consider the function

$$f(x) = \begin{cases} \dfrac{3-x^2}{2}, & 0 \le x \le 1; \\ \dfrac{1}{x}, & x > 1. \end{cases}$$

Find all points c, $0 < c < 2$, guaranteed by the Mean Value Theorem for the line segment $[0,2]$. Explain in detail why the Mean Value Theorem is applicable.

5.151. Using the Mean Value Theorem, prove that

$$\frac{a-b}{a} < \ln\frac{a}{b} < \frac{a-b}{b},$$

where $0 < b < a$.

5.152. Using the Mean Value Theorem, prove that

$$\frac{\alpha-\beta}{\cos^2\beta} \le \tan\alpha - \tan\beta \le \frac{\alpha-\beta}{\cos^2\alpha},$$

where $0 < \beta \le \alpha < \dfrac{\pi}{2}$.

5.153. Show that the functions $f(x) = \tan^{-1}\left(\frac{1+x}{1-x}\right)$ and $g(x) = \tan^{-1}x$ have the same derivative for $x \ne 1$. Find the relationship between these functions.

5.154. Using the Mean Value Theorem, prove that

$$a)\ 2\tan^{-1}x + \sin^{-1}\left(\frac{2x}{1+x^2}\right) = \pi\,\mathrm{sign}\,x,\ \ |x| \ge 1;$$

$$b)\ 3\cos^{-1}x - \cos^{-1}(3x-4x^3) = \pi,\ \ |x| < 0.5.$$

5.155. Prove that if $f(x)$ is continuous and differentiable on $[a,b]$ and $f(x)$ is not a linear function, then there is at least one point c on the open interval (a,b) such that

$$\left|f'(c)\right| > \left|\frac{f(b)-f(a)}{b-a}\right|.$$

Chapter 6.
APPLICATIONS OF THE DERIVATIVE

6.1 L'Hospital's Rule

One of the most important methods for calculating limits is **L'Hospital's rule**.

I. The indeterminate forms $\left(\frac{0}{0}\right)$ and $\left(\frac{\infty}{\infty}\right)$.

If $\lim\limits_{x\to a}\frac{f(x)}{g(x)}$ cannot be found directly, such as when (1) $\lim\limits_{x\to a} f(x)=0$ and $\lim\limits_{x\to a} g(x)=0$, giving rise to the indeterminate form $\left(\frac{0}{0}\right)$, or (2) $\lim\limits_{x\to a} f(x)=\infty$ and $\lim\limits_{x\to a} g(x)=\infty$, giving rise to the indeterminate form $\left(\frac{\infty}{\infty}\right)$, then

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)},$$

assuming that the second limit exists or equals infinity. If necessary, L'Hospital's rule can be used several times in succession. Note also that L'Hospital's rule remains valid for $x\to\infty$.

Remember that L'Hospital's rule can only be used for indeterminate forms!

Example 6.1. Find a) $\lim\limits_{x\to 1}\frac{x^3-1}{\ln x}$; b) $\lim\limits_{x\to+\infty}\frac{\frac{\pi}{2}-\tan^{-1}x}{\ln(1+\frac{1}{x^2})}$; c) $\lim\limits_{x\to+\infty}\frac{x^2}{e^x}$.

Solution. a) Since $\lim\limits_{x\to 1}(x^3-1)=0$ and $\lim\limits_{x\to 1}\ln x=0$, we can use L'Hospitals' rule:

$$\lim_{x\to 1}\frac{x^3-1}{\ln x}=\lim_{x\to 1}\frac{3x^2}{\frac{1}{x}}=\lim_{x\to 1}3x^3=3.$$

b) Since $\lim\limits_{x\to+\infty}\tan^{-1}x=\frac{\pi}{2}$, it is again necessary to use L'Hospital's rule:

$$\lim_{x\to+\infty}\frac{\frac{\pi}{2}-\tan^{-1}x}{\ln(1+\frac{1}{x^2})}=\lim_{x\to+\infty}\frac{-\frac{1}{1+x^2}}{\frac{1}{1+1/x^2}\left(-\frac{2}{x^3}\right)}=\lim_{x\to+\infty}\frac{1}{1+x^2}\cdot\frac{x^2+1}{x^2}\cdot\frac{x^3}{2}=+\infty.$$

c) Here we have an indeterminate form of the type $\left(\frac{\infty}{\infty}\right)$, and L'Hospital's rule can be used. Note that it is necessary to use L'Hospital's rule twice:

$$\lim_{x\to+\infty}\frac{x^2}{e^x}=\lim_{x\to+\infty}\frac{2x}{e^x}=\lim_{x\to+\infty}\frac{2}{e^x}=\left[\frac{2}{+\infty}\right]=0.$$

II. The indeterminate forms $(\mathbf{0}\cdot\infty)$ and $(\infty-\infty)$.

If $\lim\limits_{x\to a} f(x)=0$ and $\lim\limits_{x\to a} g(x)=\infty$, then finding $\lim\limits_{x\to a} f(x)g(x)$ involves dealing with the indeterminate form $(0\cdot\infty)$.

If $\lim\limits_{x\to a} f(x)=+\infty$ and $\lim\limits_{x\to a} g(x)=+\infty$, then the limit $\lim\limits_{x\to a}\big(f(x)-g(x)\big)$ is also indeterminate and can be expressed as $(\infty-\infty)$. Note that this limit will also be indeterminate if $\lim\limits_{x\to a} f(x)=-\infty$ and $\lim\limits_{x\to a} g(x)=-\infty$. On the other hand, if $\lim\limits_{x\to a} f(x)=+\infty$ and $\lim\limits_{x\to a} g(x)=-\infty$, then $\lim\limits_{x\to a}\big(f(x)-g(x)\big)=(+\infty-(-\infty))=+\infty$, and this limit is *not* an indeterminate form.

These limits can be reduced to the indeterminate forms $\left(\frac{0}{0}\right)$ or $\left(\frac{\infty}{\infty}\right)$ by algebraic transformations, after which they can be calculated using L'Hospital's rule.

Example 6.2. Find the following limits:

$$\text{a) } \lim_{x\to\pi/2}\left(x-\frac{\pi}{2}\right)\tan x;\quad \text{b) } \lim_{x\to 1}\left(\frac{1}{\ln x}-\frac{1}{x-1}\right);\quad \text{c) } \lim_{x\to+\infty}\left(e^x-x^2\right).$$

Solution. a) It is enough to use trigonometric transformations here:

$$\lim_{x\to\pi/2}\left(x-\frac{\pi}{2}\right)\tan x=\lim_{x\to\pi/2}\frac{x-\frac{\pi}{2}}{\cot x}=\lim_{x\to\pi/2}\frac{1}{-\frac{1}{\sin^2 x}}=-\lim_{x\to\pi/2}\sin^2 x=-1.$$

b) Simplifying,

$$\lim_{x\to 1}\left(\frac{1}{\ln x}-\frac{1}{x-1}\right)=\lim_{x\to 1}\frac{x-1-\ln x}{(x-1)\ln x}=\lim_{x\to 1}\frac{1-\frac{1}{x}}{\ln x+\frac{x-1}{x}}=\lim_{x\to 1}\frac{x-1}{x\ln x+x-1}.$$

Using L'Hospital's rule a second time,

$$\lim_{x\to 1}\frac{x-1}{x\ln x+x-1}=\lim_{x\to 1}\frac{1}{\ln x+1+1}=\frac{1}{2}.$$

Therefore, $\lim\limits_{x\to 1}\left(\frac{1}{\ln x}-\frac{1}{x-1}\right)=\frac{1}{2}$.

c) We will use algebraic transformations to find this limit:

$$\lim_{x\to+\infty}\left(e^x-x^2\right)=\lim_{x\to+\infty}e^x\left(1-\frac{x^2}{e^x}\right)$$

The limit of the expression in parenthesis is

$$\lim_{x\to+\infty}\left(1-\frac{x^2}{e^x}\right)=1-\lim_{x\to+\infty}\frac{2x}{e^x}=1-\lim_{x\to+\infty}\frac{2}{e^x}=1-0=1.$$

Therefore, since $\lim\limits_{x\to+\infty}e^x=+\infty$, $\lim\limits_{x\to+\infty}e^x\left(1-\frac{x^2}{e^x}\right)=[+\infty\cdot 1]=+\infty$.

III. The indeterminate form (1^∞).

If $\lim_{x\to a} f(x) = 1$ and $\lim_{x\to a} g(x) = \infty$, then finding $\lim_{x\to a} f(x)^{g(x)}$ involves dealing with the indeterminate form (1^∞).

This problem can be reduced to the indeterminate form $(0\cdot\infty)$ (which in its turn can be reduced to the form $(\frac{0}{0})$ or $(\frac{\infty}{\infty})$) by using the properties of the exponential function:

$$f(x)^{g(x)} = e^{\ln f(x)^{g(x)}} = e^{g(x)\ln f(x)}.$$

(The function $g(x)$ approaches infinity, while $f(x)$ approaches 1 and so $\ln f(x)$ approaches 0.)

Example 6.3. Find $\lim_{x\to 0}\left(\frac{e^x-1}{x}\right)^{\frac{1}{\sin x}}$.

Solution. This expression is an indeterminate form of the type (1^∞) at $x = 0$, because $\lim_{x\to 0}\frac{e^x-1}{x} = 1$ and $\lim_{x\to 0}\frac{1}{\sin x} = \infty$.

Transforming the expression gives

$$\lim_{x\to 0}\left(\frac{e^x-1}{x}\right)^{\frac{1}{\sin x}} = \lim_{x\to 0} e^{\frac{\ln\left(\frac{e^x-1}{x}\right)}{\sin x}} = e^{\lim_{x\to 0}\frac{\ln(e^x-1)-\ln x}{\sin x}}.$$

Thus, the problem reduces to finding the limit

$$\lim_{x\to 0}\frac{\ln(e^x-1)-\ln x}{\sin x} = \lim_{x\to 0}\frac{\frac{e^x}{e^x-1}-\frac{1}{x}}{\cos x} = \lim_{x\to 0}\frac{xe^x-e^x+1}{x(e^x-1)} =$$

$$= \lim_{x\to 0}\frac{e^x+xe^x-e^x}{e^x+xe^x-1} = \lim_{x\to 0}\frac{xe^x}{xe^x+e^x-1} = \lim_{x\to 0}\frac{1}{1+\frac{e^x-1}{xe^x}} = \frac{1}{1+1} = \frac{1}{2}.$$

Therefore, the answer is

$$\lim_{x\to 0}\left(\frac{e^x-1}{x}\right)^{\frac{1}{\sin x}} = \sqrt{e}.$$

Find the following limits using L'Hospital's rule.

6.1. $\lim_{x\to 1}\frac{x^3-3x^2+2}{x^3-4x^2+3}$

6.2. $\lim_{x\to 0}\frac{\sin ax}{\sin bx}$

6.3. $\lim_{x\to 0}\frac{\tan x - x}{x-\sin x}$

6.4. $\lim_{x\to\pi/4}\frac{\sqrt[3]{\tan x}-1}{2\sin^2 x-1}$

6.5. $\lim_{x\to a}\frac{a^x-x^a}{x-a}$

6.6. $\lim_{x\to 0}\frac{\ln(\sin ax)}{\ln(\sin bx)}$

6.7. $\lim_{x\to 1}\frac{x^2-1+\ln x}{e^x-e}$

6.8. $\lim_{x\to 1}\frac{\ln(x-1)}{\cot \pi x}$

6.9. $\lim_{x\to 0}\frac{e^x-e^{-x}}{\ln(1+x)}$

6.10. $\lim_{x\to 0}\frac{(a+x)^x-a^x}{x^2}$

6.11. $\lim\limits_{x\to\infty} \dfrac{x^n}{e^{ax}} \quad (a,n>0)$

6.12. $\lim\limits_{x\to 0^+} x^{\varepsilon}\ln x \quad (\varepsilon>0)$

6.13. $\lim\limits_{x\to 0} \dfrac{x-\sin x}{x^3}$

6.14. $\lim\limits_{x\to 0} \left(\dfrac{1}{x}-\dfrac{1}{e^x-1}\right)$

6.15. $\lim\limits_{x\to 1} \left(\dfrac{1}{\ln x}-\dfrac{1}{x-1}\right)$

6.16. $\lim\limits_{x\to 0} \dfrac{e^{3x}-3x-1}{\sin^2 5x}$

6.17. $\lim\limits_{x\to 0} \dfrac{a^x-a^{\sin x}}{x^3}$

6.18. $\lim\limits_{x\to 0} \left(\dfrac{1}{x^2}-\cot^2 x\right)$

6.19. $\lim\limits_{x\to 0} \dfrac{\sin 3x-3xe^x+3x^2}{\tan^{-1}x-\sin x-x^3/6}$

6.20. $\lim\limits_{x\to 0} (1-\cos x)\cot^2 x$

6.21. $\lim\limits_{x\to\infty} (x+2^x)^{1/x}$

6.22. $\lim\limits_{x\to 0} \left(\dfrac{\tan x}{x}\right)^{1/x^2}$

6.23. $\lim\limits_{x\to\infty} \left(\tan\dfrac{\pi x}{2x+1}\right)^{1/x}$

6.24. $\lim\limits_{x\to 0} \dfrac{x(e^x+1)-2(e^x-1)}{x^3}$

6.25. $\lim\limits_{x\to\infty} \dfrac{\pi-2\tan^{-1}x}{e^{3/x}-1}$

6.26. $\lim\limits_{x\to 0} \left(\dfrac{\sin^{-1}x}{x}\right)^{1/x^2}$

6.27. $\lim\limits_{x\to 0} \left(\dfrac{(1+x)^{1/x}}{e}\right)^{1/x}$

6.28. $\lim\limits_{x\to 0} \left(\dfrac{2\cos x}{e^x+e^{-x}}\right)^{1/x^2}$

Check that L'Hospital's rule is not applicable to the following limits:

6.29. $\lim\limits_{x\to 0} \dfrac{x^2\sin\frac{1}{x}}{\sin x}$

6.30. $\lim\limits_{x\to\infty} \dfrac{x-\sin x}{x+\sin x}$

6.31. Find the value of the limit

$$\lim_{h\to 0}\frac{f(x-h)-2f(x)+f(x+h)}{h^2},$$

assuming it exists.

6.2 Monotonicity

Definition. The function $f(x)$ is **strictly increasing** on (a,b), if for any points x_1 and x_2 $(x_1<x_2)$ on this interval we have $f(x_1)<f(x_2)$.

Definition. The function $f(x)$ is **strictly decreasing** on (a,b), if for any points x_1 and x_2 $(x_1<x_2)$ on this interval we have $f(x_1)>f(x_2)$.

Theorem. If the differentiable function $f(x)$ is strictly increasing on (a,b), then $f'(x)\ge 0$ for all $x\in(a,b)$.

Theorem. If $f'(x)>0$ for all $x\in(a,b)$, then $f(x)$ is strictly increasing on (a,b).

Analogous theorems can be proven for decreasing functions.

Important: Note that a strictly increasing function can have a zero derivative at isolated points. This behavior is exhibited, for instance, by the function $f(x)=x^3$. This function is strictly increasing for all x, and yet $f'(0)=0$.

The same is true for strictly decreasing functions.

Example 6.4. Find the intervals on which the function $f(x)=3x^4-4x^3-12x^2+2$ is increasing and decreasing.

Solution. First find the derivative:

$$f'(x)=12x^3-12x^2-24x=12x(x^2-x-1)=12x(x+1)(x-2).$$

Critical points: $f'(x)=0$ or $f'(x)\ \nexists$:

$$12x(x+1)(x-2)=0 \quad\Rightarrow\quad x_1=0;\quad x_2=-1,\quad x_3=2.$$

Check the sign of the derivative:

x		-1		0		2	
$f'(x)$	$-$	0	$+$	0	$-$	0	$+$
$f(x)$	↘		↗		↘		↗

We see that $f'(x)$ is positive for $x\in(-1,0)$ and for $x\in(2,\infty)$, while $f'(x)$ is negative for $x\in(-\infty,-1)$ and $x\in(0,2)$. Therefore, $f(x)$ is increasing for $x\in(-1,0)\cup(2,+\infty)$ and decreasing for $x\in(-\infty,-1)\cup(0,2)$.

Find the intervals on which the functions are strictly increasing or decreasing.

6.32. $f(x)=3x-x^3$

6.33. $f(x)=x^4-2x^2-5$

6.34. $f(x)=(x-2)^5(2x+1)^4$

6.35. $f(x)=\dfrac{2x}{1+x^2}$

6.36. $f(x)=\dfrac{\sqrt{x}}{x+100}$

6.37. $f(x)=x-e^x$

6.38. $f(x)=(x+1)e^{-x}$

6.39. $f(x)=x^2e^{-x}$

6.40. $f(x)=x^2-\ln(x^2)$

6.41. $f(x)=\dfrac{x}{\ln x}$

6.42. $f(x)=x+\cos x$

6.43. $f(x)=\dfrac{x^2}{2^x}$

6.44. $f(x)=2\sin x+\cos 2x\ (0\le x\le 2\pi)$

6.45. $f(x)=x+|\sin 2x|$

6.3 Related rates

Problems in related rates deal with the change of various quantities (physical or geometrical) with *time*.

Definition. The **rate of change** of a quantity is the derivative of that quantity with respect to time.

Always remember to include units of measurement!!

Position, Velocity, Acceleration (P.V.A.)

Definition. The **position** of a particle moving along the x-axis is given by the function $x(t)$.

Definition. The **velocity** of a particle moving along the x-axis is given by $\frac{dx}{dt}$.

Definition. The **speed** of a particle moving along the x-axis is given by $\left|\frac{dx}{dt}\right|$.

Definition. The **acceleration** of a particle moving along the x-axis is given by $\frac{d^2x}{dt^2}$ or $\frac{dv}{dt}$.

Analogous definitions are valid for movement along the y-axis.

Example 6.5. A rectangle has sides of 20 and 40 inches, respectively. The larger sides of the rectangle begin to shrink at a rate of 2 inches per second. How fast is the area of the rectangle changing at this moment?

Solution. Let a be the length of the smaller sides of the rectangle and b be the length of the larger sides. Since b is *decreasing*, we have

$$\frac{db}{dt} = -2. \quad \text{(inches per second)}$$

Note that a remains constant, so

$$\frac{da}{dt} = 0. \quad \text{(inches per second)}$$

The area of the rectangle is given by $A = ab$, so its rate of change is given by

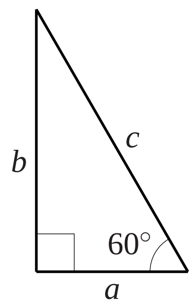
$$\frac{dA}{dt} = \frac{d}{dt}(ab) = \frac{da}{dt}b + a\frac{db}{dt} = -2a. \quad (\text{in}^2 \text{ per second})$$

At the instant when $a = 20$ inches and $b = 40$ inches,

$$\left.\frac{dA}{dt}\right|_{\substack{a=20\\ b=40}} = -40. \quad (\text{in}^2 \text{ per second})$$

Example 6.6. The hypotenuse of a right triangle is increasing at the rate of 4 inches per minute, while all the angles in the triangle remain constant. At the instant when the sides of the triangle are 10, $10\sqrt{3}$ and 20 inches, determine a) how fast the perimeter of the triangle is changing; b) how fast the area of the triangle is changing.

Solution. Let a, b and c be the lengths of the sides of the triangle, as shown below. Note that the angles in the triangle are 30°, 60° and 90°.



According to the problem, $\frac{dc}{dt} = 4$ inches per minute. The perimeter of the triangle equals $P = a+b+c$, so

$$\frac{dP}{dt} = \frac{da}{dt} + \frac{db}{dt} + \frac{dc}{dt}. \quad \text{(inches per minute)}$$

Find the rate of change of a:

$$a = c \cdot \cos(60^\circ) = \frac{c}{2} \quad \Rightarrow \quad \frac{da}{dt} = \frac{1}{2}\frac{dc}{dt} = 2. \quad \text{(inches per minute)}$$

Find the rate of change of b:

$$b = c \cdot \sin(60^\circ) = \frac{\sqrt{3}}{2}c \quad \Rightarrow \quad \frac{db}{dt} = \frac{\sqrt{3}}{2}\frac{dc}{dt} = 2\sqrt{3}. \quad \text{(inches per minute)}$$

Therefore,

$$\frac{dP}{dt} = 2 + 2\sqrt{3} + 4 = 6 + 2\sqrt{3}. \quad \text{(inches per minute)}$$

Note that the rate of change of the perimeter is constant.

It is easiest to find the rate of change of the area of the triangle by using the formula $A = \frac{1}{2}ab$, so

$$\frac{dA}{dt} = \frac{1}{2}\left(\frac{da}{dt}b + a\frac{db}{dt}\right) = b + \sqrt{3}a. \quad (\text{in}^2 \text{ per minute})$$

Therefore,

$$\left.\frac{dA}{dt}\right|_{\substack{a=10 \\ b=10\sqrt{3}}} = 20\sqrt{3}. \quad (\text{in}^2 \text{ per minute})$$

Example 6.7. A particle is moving along the curve $y = x^2$ so that its x-coordinate is increasing at a constant rate of 3 units per second. How fast is its y-coordinate changing at the instant when $x = 4$?

Solution. Using the chain rule, $y = y\big(x(t)\big)$,

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt} = 2x\frac{dx}{dt} = 6x. \quad \text{(units per second)}$$

Therefore, when $x = 4$ the y-coordinate is increasing at a rate of

$$\left.\frac{dy}{dt}\right|_{x=4} = 24. \quad \text{(units per second)}$$

6.46. The radius of a circle is decreasing at a constant rate of 0.1 centimeter per second. In terms of the circumference C, what is the rate of change of the area of the circle, in square centimeters per second?

6.47. A ship is 400 miles directly south of Tahiti and is sailing south at 20 miles per hour. Another ship is 300 miles east of Tahiti and is sailing west at 15 miles per hour. At what rate is the distance between the ships changing?

6.48. Two steamships leave a port at the same time. The first one moves straight north at a speed of 20 miles per hour, while the second moves west at a speed of 25 miles per hour. How fast is the distance between them changing one hour later?

6.49. A man on a pier pulls in a rope attached to a small boat at the rate of 1 foot per second. If his hands are 10 feet above the place where the rope is attached, how fast is the boat approaching the pier when there is 20 feet of rope out?

6.50. A cylindrical swimming pool is being filled from a fire hose at the rate of 5 cubic feet per second. If the pool is 40 feet across, how fast is the water level increasing when the pool is one third full?

6.51. A cube is contracting so that its surface area is decreasing at the constant rate of $72\frac{\text{in}^2}{\text{sec}}$. Determine how fast the volume is changing, in cubic inches per second, at the instant when the surface area is 54ft^2. (1 foot equals 12 inches.)

6.52. A sphere is increasing in volume at the rate of $16\pi\frac{\text{cm}^3}{\text{sec}}$. At what rate is its radius changing when the radius is 3cm? (The volume of a sphere is given by $V = \frac{4}{3}\pi R^3$.)

6.53. How fast are the area and diagonal of a rectangle changing when one side is 20ft and the second is 15ft, if the first side is shrinking at a rate of $1\frac{\text{ft}}{\text{sec}}$ and the length of the second side is increasing at a rate of $2\frac{\text{ft}}{\text{sec}}$?

6.54. A bird is flying in a straight line east at $25\frac{\text{ft}}{\text{sec}}$. An observer is 15 ft to the south of the spot where the bird began to fly. Consider the angle formed by the line connecting the bird and the observer and the line connecting the bird and its initial position. Find the rate of change (in radians per second) of the angle when the bird has flown 20 ft.

6.55. Cement is poured so that it continuously forms a conical pile, the height of which is twice the radius of the base. If the cement is being poured at the rate of 12 cubic feet per second, how fast is the height of the pile changing when it is 4 feet high? (The volume of a cone is given by $V = \frac{\pi}{3}R^2 h$.)

6.56. The resistance of a parallel electric circuit R is given by

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}.$$

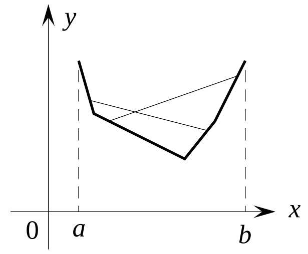
If R_1 is increasing at a rate of 0.5 ohm per second and R_2 is decreasing at a rate of 0.4 ohm per second, how fast is R changing when $R_1 = 200\text{ohm}$ and $R_2 = 300\text{ohm}$?

6.57. A clay cylinder is being compressed so that its height is changing at the rate of 4 milimeters per second, and its diameter is increasing at the rate of 2 milimeters per second. Find the rate of change of the area of the horizontal cross-section of the cylinder when its height is 1 centimeter.

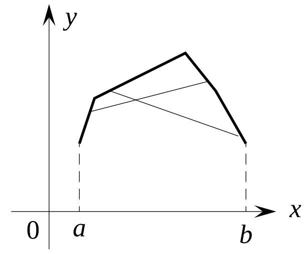
6.4 Convexity and concavity

The geometric concepts of convexity and concavity

The function $f(x)$ is **concave up** or **convex** on (a,b) if the graph of $f(x)$ is located *not higher than* any of its secant lines.



The function $f(x)$ is **concave down** or **concave** on (a,b) if the graph of $f(x)$ is located *not lower than* any of its secant lines.



Note that these concepts do not require the function to have a second derivative, or even, for that matter, to be continuous!

Algebraic formulation of concavity and convexity

Definition. A function is **convex** on (a,b) if the inequality

$$f\big(\alpha x+(1-\alpha)y\big) \le \alpha f(x)+(1-\alpha)f(y)$$

is satisfied for any two points x and y from (a,b) and any α in $[0,1]$.

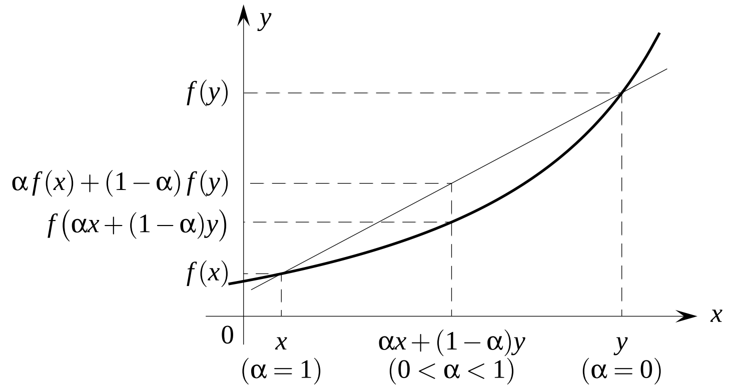
Definition. A function is **strictly convex** on (a,b) if the inequality

$$f\big(\alpha x+(1-\alpha)y\big) < \alpha f(x)+(1-\alpha)f(y)$$

is satisfied for any two points x and y from (a,b) and any α in $(0,1)$.

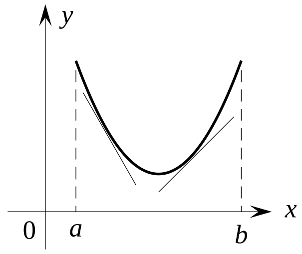
Analogous definitions hold for (strictly) concave functions.

The expression $\alpha x+(1-\alpha)y$ for $\alpha\in[0,1]$ is simply the interval $[x,y]$; for $\alpha=0$ this expression equals y, and for $\alpha=1$ it equals x. In the same way, the expression $\alpha f(x)+(1-\alpha)f(y)$ for $\alpha\in[0,1]$ is simply the interval $[f(x),f(y)]$. The relationship between the geometrical intuition and the algebraic definition is shown on the figure below.

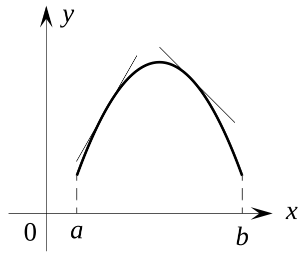


Graphical interpretation for smooth functions

A function that is strictly concave up on (a,b) will have a graph that is located *above* all of its tangent lines.



A function that is strictly concave down on (a,b) will have a graph that is located *below* all of its tangent lines.



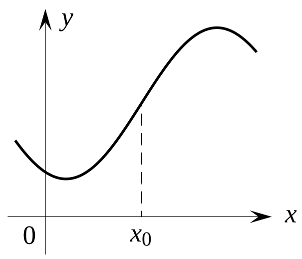
Theorem. If the twice-differentiable function $f(x)$ is concave up on (a,b), then $f''(x) \geq 0$ for all $x \in (a,b)$.

Theorem. If $f''(x) > 0$ for all $x \in (a,b)$, then $f(x)$ is concave up on (a,b).

Analogous theorems can be proven for functions that are concave down.

Important: Note that a function that is concave up or concave down can have a zero second derivative at isolated points. This behavior is exhibited, for instance, by the function $f(x) = x^4$. This function is concave up for all x, and yet $f''(0) = 0$.

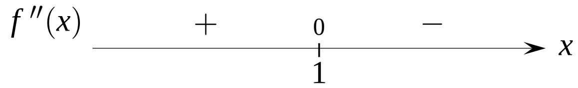
Definition. If $f(x)$ is continuous at $x = x_0$ and its concavity changes at that point, i.e. $f''(x)$ changes its sign at $x = x_0$, then $x = x_0$ is a **point of inflection** of $f(x)$.



Example 6.8. Find the intervals on which the function $f(x) = 3x^2 - x^3$ is concave upward and concave downward, and determine the inflection points.

Solution. First find the second derivative: $f'(x) = 6x - 3x^2$; $f''(x) = 6 - 6x$. We see that the second derivative equals zero at $x = 1$.

Now check the sign of the second derivative:



Therefore, the graph of $f(x)$ is concave up for $x \in (-\infty, 1)$ and concave down for $x \in (1, +\infty)$; at $x = 1$ this function has an inflection point.

Find the intervals on which the function is concave upward and concave downward, and determine the inflection points.

6.58. $f(x) = x^3 - 5x^2 + 3x - 5$

6.59. $f(x) = (x+2)^6 + 2x + 4$

6.60. $f(x) = x^4 - 12x^3 + 48x^2 - 50$

6.61. $f(x) = x + x^{5/3}$

6.62. $f(x) = \sqrt{1+x^2}$

6.63. $f(x) = (x+1)^4 + e^x$

6.64. $f(x) = \dfrac{x^3}{x^2+3a^2}$ $(a > 0)$

6.65. $f(x) = \ln(1+x^2)$

6.66. $f(x) = e^{-x^2}$

6.67. $f(x) = x^x$ $(x > 0)$

6.68. $f(x) = \ln \sin x$

6.69. Show that the function

$$y = \frac{x+1}{x^2+1}$$

has three inflection points, all of which lie on the same line.

6.70. Show that the inflection points of the function $y = x\sin x$ lie on the curve

$$y^2(4+x^2) = 4x^2.$$

6.71. Show that the graphs of $y = \pm e^{-x}$ and $y = e^{-x}\sin x$ have the same tangent lines at the inflection points of $y = e^{-x}\sin x$.

6.72. For what value of h will the probability density curve

$$y = \frac{h}{\sqrt{\pi}} e^{-h^2x^2}, \ h > 0$$

have inflection points at $x = \pm\sigma$?

6.73. Find the values of a and b such that the function $y = ax^3 + bx^2$ has an inflection point at $(1,3)$.

6.74. For what values of a will the function $y = e^x + ax^3$ have inflection points?

6.75. Prove that there must be at least one inflection point between two extreme points of any twice-differentiable function.

6.76. Let $f(x)$ be a function satisfying the following conditions:

- $f''(x)$ is continuous for all x;
- $f(x)$ is constant on the interval $x < 0$;
- $f(1) = 1$ and $f(2) = 2$.

a) Show that there is point a such that $f'(a) = 1$. b) Show that for all k, $0 < k \le \frac{1}{2}$, there exists c such that $f''(c) = k$.

6.5 Optimization

Definition. The function $f(x)$ has a **local maximum** at $x = x_0$ if there exists an interval (a,b) containing x_0 such that $f(x_0) > f(x)$ for all other points in (a,b).

An analogous definition can be given for a local minimum. Local extremes (minimums and maximums) are also sometimes called **relative** extremes.

Note that this definition is *not* applicable to the endpoints of the domain. This definition demands that the function is defined on an *open* interval containing the local extreme; this assumption will not work if the point being considered is an endpoint.

Definition. If $f(x_0)$ exists and the derivative of $f(x)$ equals zero or is nonexistent at $x = x_0$, then x_0 is a **critical point** of $f(x)$.

Theorem (Necessary condition for the existence of a local extreme)

If $f(x)$ attains a local maximum or minimum at $x = x_0$ and $f'(x)$ exists at $x = x_0$, then $f'(x_0) = 0$.

Theorem (First-order sufficient condition for the existence of a local extreme)

If x_0 is a critical point and $f'(x)$ changes its sign at $x = x_0$, then $f(x_0)$ is either a local minimum or a local maximum.

Theorem (Second-order sufficient condition for the existence of a local extreme)

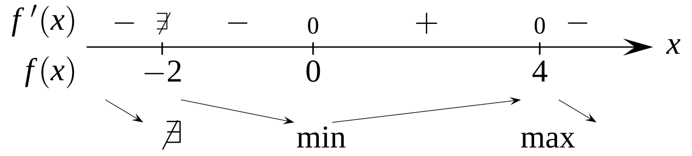
Let $f'(x_0) = 0$. If $f''(x_0) > 0$, then $f(x_0)$ is a local minimum; if $f''(x_0) < 0$, then $f(x_0)$ is a local maximum. If $f''(x_0) = 0$ or does not exist, then the second-order condition cannot be used.

Example 6.9. Find all local minimums and maximums of the function $f(x) = \dfrac{x^{2/3}}{x+2}$ using the first order sufficient condition.

Solution. This function exists for all $x \neq -2$. The derivative is

$$f'(x) = \frac{\frac{2}{3}x^{-1/3}(x+2) - x^{2/3}}{(x+2)^2} = \frac{2x+4-3x}{3x^{1/3}(x+2)^2} = \frac{4-x}{3x^{1/3}(x+2)^2}$$

The critical points of this function are $x_1 = 0$ and $x_2 = 4$. Note that $x = -2$ is not a critical point, but should be included in the following analysis to keep track of the derivative's sign.



Since $f'(x)$ changes from negative to positive at $x=0$, f has a local minimum $f(0)=0$; as $f'(x)$ changes from positive to negative at $x=4$, f has a local maximum $f(4)=\sqrt[3]{2}/3$.

Example 6.10. Find all local minimums and maximums of the function $f(x)=x^3-3x^2+1$ using the second order sufficient condition.

Solution. The derivative of $f(x)$ is

$$f'(x)=3x^2-6x.$$

Critical points are $x=0$ and $x=2$. The second derivative is

$$f''(x)=6x-6.$$

At $x=0$ we have $f''(0)=-6<0$, so $f(0)=1$ is a local maximum. At $x=2$ we have $f''(2)=6>0$, so $f(2)=-3$ is a local minimum.

Find all local minimums and maximums of the following functions.

6.77. $f(x)=2x^2-x^4$

6.78. $f(x)=\dfrac{2x}{1+x^2}$

6.79. $f(x)=(x-2)^{2/3}(2x+1)$

6.80. $f(x)=\dfrac{x^2-3x+2}{x^2+2x+1}$

6.81. $f(x)=\dfrac{3x^2+4x+4}{x^2+x+1}$

6.82. $f(x)=x\sqrt[3]{x-1}$

6.83. $f(x)=\sqrt{2x-x^2}$

6.84. $f(x)=\dfrac{1}{\ln(x^4+4x^3+30)}$

6.85. $f(x)=\tan^{-1}x-\dfrac{1}{2}\ln(1+x^2)$

Definition. The function $f(x)$ has a **global maximum** at $x=x_0$ if $f(x_0)\geq f(x)$ for all x in the domain of $f(x)$.

An analogous definition can be given for a global minimum. Global extremes are also sometimes called **absolute** extremes.

An enormously important theorem for finding global extremes is the *Extreme Value Theorem* (see page 25), which states that a continuous function on a closed interval must attain its minimum and maximum values. It is important to remember that global extremes are often found at the endpoints of the closed interval.

Theorem (Sufficient condition for a global extreme)

If $f'(x_0)=0$ and $f''(x)>0$ ($f''(x)<0$) for all x, then $f(x_0)$ is a global minimum (global maximum).

In other words, if the function is concave up everywhere on the interval and has a critical point, then it has a global minimum at the critical point and a global maximum at one of the endpoints. If the function is concave down everywhere on the interval and has a critical point, then it has a global maximum at its critical point and a global minimum at one of the endpoints.

Example 6.11. Find the global extremes of $f(x) = x^2 - 4x$ on the interval $[-2,5]$.

Solution. First find the critical points:

$$f'(x) = 2x - 4; \quad f'(x) = 0 \quad \Leftrightarrow \quad x = 2.$$

The second derivative is $f''(x) = 2$, so $f''(x) > 0$ for all x. Therefore, $f(2) = -4$ is the global minimum of $f(x)$.

In order to find the global maximum, we will consider the values of $f(x)$ at the endpoints: $f(-2) = 12$, $f(5) = 5$. Therefore, the global maximum is $f(-2) = 12$.

In the general case, when $f(x)$ is neither concave up nor concave down everywhere on the interval, the sufficient condition for a global extreme can not be used. It is necessary to find all local extremes and then compare them to the values of the function at the endpoints of the interval, i.e. at $x = a$ and $x = b$.

Note that a continuous function on an *open* interval might have only local extremes and no global extremes. In order to find the global extremes of a continuous function on an open interval, it is necessary to find all local extremes and investigate the behavior of the function as x approaches the endpoints of the interval.

Example 6.12. Find the global extremes of $f(x) = x^3 - 3x$ on the interval $[-3, \sqrt{3}]$.

Solution. First find the critical points:

$$f'(x) = 3x^2 - 3; \quad f'(x) = 0 \quad \Leftrightarrow \quad x = \pm 1.$$

Use the first derivative test to find the local extremes:

								x
$f'(x)$		$+$	0	$-$	0	$+$		
$f(x)$	-3	↗	-1	↘	1	↗	$\sqrt{3}$	
			max		min			

As $f'(x)$ changes from positive to negative at $x = -1$, f has a local maximum $f(-1) = 2$; as $f'(x)$ changes from negative to positive at $x = 1$, f has a local minimum $f(1) = -2$. The values of $f(x)$ at the endpoints are $f(-3) = -18$ and $f(\sqrt{3}) = 0$. Therefore, the absolute minimum of $f(x)$ on the interval $[-3, \sqrt{3}]$ is at -3 and equals -18, while the absolute maximum of $f(x)$ is at -1 and equals 2.

Find the global minimum and maximum values of the following functions on the given interval.

6.86. $f(x) = x^2 - 4x + 6,\ x \in [-3, 10]$ **6.87.** $f(x) = x^2 - 5x + 7,\ x \in [-1, 3]$

6.88. $f(x) = x^3 - 6x^2 + 9x + 2,\ x \in [0, 4]$

6.89. $f(x) = x^5 - 5x^4 + 5x^3 + 1,$ $x \in [-1, 2]$

6.90. $f(x) = \dfrac{x-1}{x+1},\ x \in [0, 4]$ **6.91.** $f(x) = \dfrac{1 - x + x^2}{1 + x - x^2},\ x \in [0, 1]$

6.92. $f(x) = \sqrt{5 - 4x},\ x \in [-1, 1]$ **6.93.** $f(x) = \sin 2x - x,\ x \in \left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$

6.94. $f(x) = \ln(x^2+2x+4)$, $x \in [-4,3]$

6.95. $f(x) = \left|x^2-3x+2\right|$, $x \in [-10,10]$ **6.96.** $f(x) = \sqrt[3]{(x^2-2x)^2}$, $x \in [0,3]$

6.97. $f(x) = \tan^{-1}\left(\dfrac{1-x}{1+x}\right)$, $x \in [0,1]$ **6.98.** $f(x) = x^x$, $x \in [0.1,\infty)$

6.99. Find two nonnegative numbers whose sum is 9 so that the product of one number and the square of the other number is the maximum possible.

6.100. Determine the maximum acceleration attained on the interval $0 \le t \le 3$ by a particle whose velocity in meters per second is given by $v(t) = t^3 - 3t^2 + 12t + 4$. Indicate units of measure.

6.101. Of all lines tangent to the graph of $y = \frac{6}{x^2+3}$, find the tangent lines with the minimum and maximum slope.

6.102. The area of a rectangle is 25 square feet. What are the lengths of its sides, if its perimeter is minimal?

6.103. The sum of the height and radius of a closed cylinder is equal to 20 inches. Find a) the maximum possible volume and b) the maximum possible surface area of the cylinder.

6.104. Find the acute angles of the right triangle with the maximum possible area, if the sum of one leg and the hypotenuse is constant.

6.105. Find the maximum volume of a right cone that has a side length of L.

6.106. The daily cost of running a ship equals \$800 plus one twentieth of the cube of the ship's speed. What speed will minimize the cost of running the ship?

6.107. A truck has a minimum speed of 10 m.p.h. in high gear. When traveling at x m.p.h, the truck burns diesel at the rate of $\frac{1}{3}\left(\frac{900}{x}+x\right)\frac{\text{gal}}{\text{mile}}$. The truck cannot go faster than 50 m.p.h. If diesel is \$2 per gallon, find a) the steady speed that will minimize the cost of fuel for a 500 mile trip; b) the steady speed that will minimize the cost of a 500 mile trip if the driver is paid \$15 an hour.

6.108. A closed rectangular container with a square base is to be made from two different materials. The material for the base costs \$5 per square meter, while the material for the other five sides costs \$1 per square meter. Find the dimensions of the container that has the largest possible volume if the total cost of materials is \$72.

6.109. There are 50 apple trees in an orchard. Each tree produces 800 apples. For each additional tree planted in the orchard, the output per tree drops by 10 apples. How many trees should be added to the existing orchard in order to maximize the total output of the trees?

6.110. A railroad company can run a train only with a minimum of 200 passengers. The fare will be \$8 and decreased by 1 cent for each person over the 200 minimum requirement. How many passengers must travel for maximum revenue?

6.111. Find the dimensions of the rectangle of the largest area that can be inscribed in the closed region bounded by the x-axis, y-axis, and the graph of $y = 8 - x^3$.

6.112. Find the distance between the point $(0,2)$ and the parabola $y = x^2/2$.

6.113. What is the area of the largest rectangle that can be inscribed in the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, if the sides of the rectangle are parallel to the x and y-axes?

6.114. Text on a page must take up exactly 24 square centimeters. The top and bottom margins must be equal to 2 centimeters, while the left and right margins must be equal to 3 centimeters. What should the dimensions of the block of text be in order to economize paper?

6.115. Text on a page must take up exactly S square centimeters. The top and bottom margins must be equal to a centimeters, while the left and right margins must be equal to b centimeters. What should the dimensions of the block of text be in order to economize paper?

6.116. A movie screen on a wall is 20 feet high and 10 feet above your eyelevel. At what distance from the front of the room should you position yourself so that your angle of viewing the screen is as large as possible?

6.117. An aircraft climbing at a constant angle of 30° above the horizontal passes directly over a radar station at an altitude of 1 kilometer. At a later instant, the radar shows that the aircraft is at a distance of 2 kilometers from the station and that this distance is increasing at the rate of 7 kilometers per minute. What is the speed of the aircraft at that instant?

6.118. The curvature of a function is defined as

$$\left| \frac{\frac{d^2 f}{dx^2}}{\left[1+\left(\frac{df}{dx}\right)^2\right]^{3/2}} \right| .$$

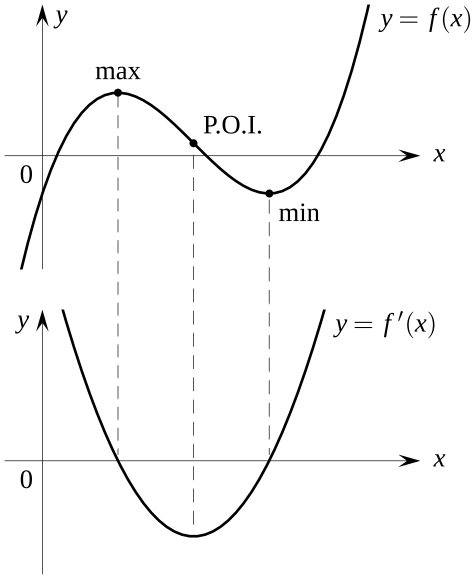
Find the value of x for which the curvature of the function $y = \ln x$ is maximal.

6.6 Function graphs

The relationship between the graphs of $f(x)$ and $f'(x)$

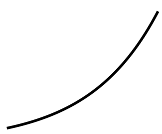
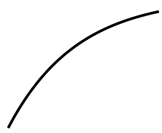
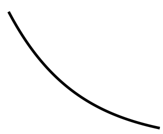
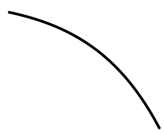
For differentiable functions, local extremes are found where the derivative changes its sign.

For twice-differentiable functions, points of inflection are where $f''(x)$ changes its sign. However, $f''(x) = \left(f'(x)\right)'$, which implies that the derivative of $f'(x)$ changes its sign; thus points of inflection of the function are found where $f'(x)$ has a local extreme.



Using $f'(x)$ and $f''(x)$ to sketch the graph of $f(x)$

There are four possible combinations of the signs of $f'(x)$ and $f''(x)$. The shape of the graph of $f(x)$ for each combination is given in the table below.

$f'(x)$	$+$	$+$	$-$	$-$
$f''(x)$	$+$	$-$	$+$	$-$
$f(x)$	 increasing, concave up	 increasing concave down	 decreasing concave up	 decreasing concave down

Thus, in order to correctly graph the function $f(x)$, it is necessary to find the intervals on which both $f'(x)$ and $f''(x)$ do not change their signs.

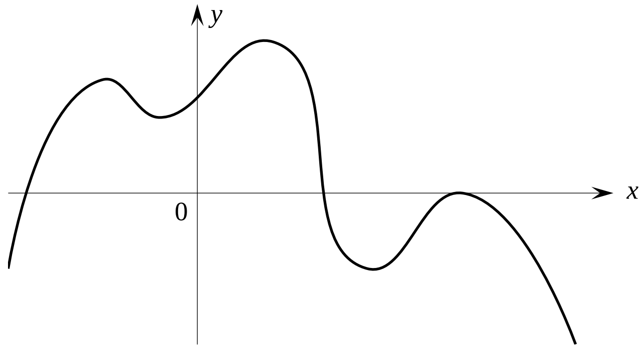
6.119. Let M_1 be the number of local minimums of a twice-differentiable function, and M_2 be its number of local maximums. If M_1 and M_2 are finite, explain why $|M_1 - M_2| \leq 1$.

6.120. Explain why a twice-differentiable function that has M_1 local minimums and M_2 local maximums must have not less than $M_1 + M_2 - 1$ points of inflection.

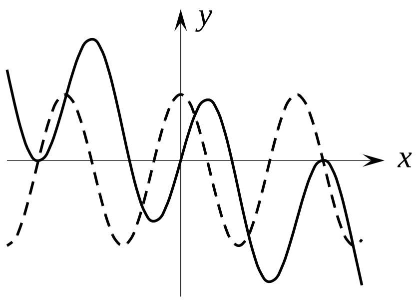
6.121. Explain why a twice-differentiable, non-linear function that has horizontal asymptotes at ∞ and $-\infty$ must have at least one point of inflection.

6.122. Sketch the graph of a twice-differentiable, non-linear function with slant asymptotes at ∞ and $-\infty$ that does not have any points of inflection.

6.123. The graph of f', the derivative of f, is given below. How many a) local minimums; b) local maximums; c) points of inflection does the graph of f have?



6.124. The graphs of two functions, $h'(x)$ and $h''(x)$, are given below. a) Which graph is which and why? b) What will $h'''(x)$ (the third derivative of $h(x)$) look like? Draw one possible graph. c) What will $h(x)$ look like? Draw one possible graph.



6.125. Let $f(x)$ be a function continuous on the interval $[-5,5]$. The first and second derivatives of $f(x)$ have the properties indicated in the table below. Draw a sketch of a possible graph of $f(x)$. Assume $f(1)=0$.

x	$[-5,-4)$	-4	$(-4,-2.5)$	-2.5	$(-2.5,-2)$
$f'(x)$	>0	DNE	<0	0	<0
$f''(x)$	>0	DNE	>0	0	<0

x	-2	$(-2,-1)$	-1	$(-1,2)$	2	$(2,5]$
$f'(x)$	<0	<0	0	>0	>0	>0
$f''(x)$	0	>0	>0	>0	0	<0

6.126. The function $g(x)$ is defined for all x except for $x=1$ and $x=7$, discontinuous at $x=3$, and its range is $\mathbb{R}$. Draw a sketch of a possible graph of $g(x)$ that satisfies the following conditions:

$$\lim_{x\to-\infty} g(x) = -1; \quad \lim_{x\to1^-} g(x) = 1; \quad \lim_{x\to1^+} g(x) = 3;$$
$$\lim_{x\to3} g(x) = -3; \quad \lim_{x\to7^-} g(x) = -\infty; \quad \lim_{x\to7^+} g(x) = \infty;$$
$$\lim_{x\to\infty} g(x) = 3.$$

An accurate graph of a function requires a complete investigation of its properties:

1. Determine the domain of $f(x)$.
2. Determine whether or not the function is even, odd, or periodic.
3. Find all discontinuities and determine their type.
4. Find all asymptotes at infinity or at a point.
5. Find the first derivative of $f(x)$. Find the critical points and determine the intervals on which $f(x)$ is monotone. Use the first derivative test to find all the coordinates of local extremes.
6. Find the second derivative of $f(x)$. Find the points where $f''(x) = 0$ or $f''(x)$ does not exist and determine the intervals on which $f(x)$ is concave up or concave down. Find the coordinates of all inflection points.
7. Find all x- and y-intercepts.
8. Sketch the graph of $f(x)$ using all available information.

Sketch accurate graphs of the following functions.

6.127. $y = \dfrac{x}{1+x^2}$

6.128. $y = \dfrac{x}{x^2-1}$

6.129. $y = \dfrac{1}{x} + 4x^2$

6.130. $y = \dfrac{(x-1)^2}{(x+1)^3}$

6.131. $y = \dfrac{x^3+2x^2+7x-3}{2x^2}$

6.132. $y = \dfrac{x}{e^x}$

6.133. $y = \ln(x^2+1)$

6.134. $y = x + \dfrac{\ln x}{x}$

6.135. $y = e^{2x-x^2}$

6.136. $y = (x+2)e^{1/x}$

6.137. $y = x - 2\tan^{-1} x$

6.138. $y = (x-1)^{2/3}(x+1)^3$

6.139. $y^2 = x^2 - x^4$

6.140. $y^2 = (1-x^2)^3$

6.141. $y = e^{\tan x}$

6.142. $y = x^2 - 4|x| + 3$

6.143. $y = \sqrt{1-e^{-x^2}}$

6.144. $y = \ln\left(x + \sqrt{x^2+1}\right)$

6.145. $y = \cos^{-1}\left(\dfrac{1-x}{1-2x}\right)$.

Chapter 7.
INFINITE SERIES

7.1 Introduction to series

Consider the sequence $a_1, a_2, \ldots, a_n, \ldots$

Definition. The **N-th partial sum** is

$$S_N = a_1 + a_2 + \ldots + a_N = \sum_{n=1}^{N} a_n,$$

and the numbers $a_1, a_2, \ldots$ are called **the terms** of the series.

Definition. The **sum** of the series is the limit

$$S = \lim_{N\to\infty} S_N = \sum_{n=1}^{\infty} a_n,$$

assuming it exists.

Definition. If the sum of a series exists, then it is **convergent**; otherwise it is **divergent**.

Properties of series

1. If $\sum_{n=1}^{\infty} a_n$ converges, then for any number k the series $\sum_{n=1}^{\infty} ka_n$ converges as well, and $\sum_{n=1}^{\infty} ka_n = k\sum_{n=1}^{\infty} a_n$.
2. If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ both converge, then the series $\sum_{n=1}^{\infty} (a_n + b_n)$ also converges, and $\sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} (a_n + b_n)$.
3. If a series converges, then it will remain convergent if a *finite* number of terms are added or deleted from the series.

Theorem (Necessary condition for the convergence of a series)

If a series is convergent, then $\lim_{n\to\infty} a_n = 0$.

Example 7.1. Consider the geometric progression with $b_1 = 2$ and $q = 0.5$. The N-th partial sum is equal to

$$S_N = b_1 \frac{1-q^N}{1-q} = 2\frac{1-0.5^N}{1-0.5} = 4\left(1-0.5^N\right).$$

This series is convergent, because

$$S = \lim_{N\to\infty} S_N = \lim_{N\to\infty} 4\left(1-0.5^N\right) = 4.$$

Example 7.2. Consider the series $\sum_{n=1}^{\infty} (\ln(n+1) - \ln n)$. It is not hard to see that

$$\begin{aligned} S_1 = a_1 &= \ln 2; \\ S_2 = a_1 + a_2 = \ln 2 + (\ln 3 - \ln 2) &= \ln 3; \\ S_3 = a_1 + a_2 + a_3 = \ln 3 + (\ln 4 - \ln 3) &= \ln 4; \\ &\dots \\ S_N &= \ln(N+1) \end{aligned}$$

Obviously, $\lim_{N\to\infty} S_N = \infty$, and the series is not convergent. This example is also interesting in that the necessary condition is nevertheless *satisfied*, i.e. we have

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} (\ln(n+1) - \ln n) = \lim_{n\to\infty} \ln\left(1+\frac{1}{n}\right) = \ln 1 = 0.$$

This example therefore shows that the necessary condition for convergence is not *sufficient*.

Explain why the following series are divergent.

7.1. $\sum_{n=1}^{\infty} (-1)^n$

7.2. $\sum_{n=1}^{\infty} \sqrt{\frac{n^2+1}{3n^2+1}}$

7.3. $\sum_{n=1}^{\infty} \frac{n^2+1}{100n^2}$

7.4. $\sum_{n=1}^{\infty} \frac{(-1)^n n}{n+1}$

7.5. $\sum_{n=1}^{\infty} n \sin\frac{1}{n}$

7.6. $\sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^n$

7.2 Positive series

Definition. If a series contains only positive terms, then it is called a **positive series**.

Definition. The series $\sum_{n=1}^{\infty} \frac{1}{n}$ is called the **harmonic series**; the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is called the **generalized harmonic series**.

Theorem (Criteria for the convergence of the generalized harmonic series)

The generalized harmonic series is convergent if $p > 1$; if $p \leq 1$, then it is divergent.

Sufficient conditions for the convergence of positive series

1. **The first comparison test**

   Consider two positive series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$, where $a_n \leq b_n$ for any n. If the series $\sum_{n=1}^{\infty} b_n$ converges, then the series $\sum_{n=1}^{\infty} a_n$ must also converge; if however $\sum_{n=1}^{\infty} a_n$ diverges, then the series $\sum_{n=1}^{\infty} b_n$ diverges as well.

2. **The second comparison test**

   Consider two positive series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$, and assume that the limit $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = k$ exists. If $k \neq 0$, then these two series are either both converge or both divergent.

   Note that if $k = 0$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ also converges; and if $k = \infty$ and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ will diverge as well.

3. **The ratio test**

   Consider the positive series $\sum_{n=1}^{\infty} a_n$ and the limit $\lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n} = l$. If $l < 1$, then the series converges; if $l > 1$, then the series diverges; if $l = 1$, then more investigation of the series is required.

4. **The root test**

   Consider the positive series $\sum_{n=1}^{\infty} a_n$ and the limit $\lim\limits_{n\to\infty} \sqrt[n]{a_n} = l$. If $l < 1$, then the series converges; if $l > 1$, then the series diverges; if $l = 1$, then more investigation of the series is required.

Example 7.3. Determine whether or not the following series are convergent:

$$\text{a) } \sum_{n=2}^{\infty} \frac{1}{n^{3/2} \ln n} \qquad \text{b) } \sum_{n=1}^{\infty} \frac{n^2+1}{n^4+n-1}; \qquad \text{c) } \sum_{n=1}^{\infty} \frac{n+1}{n^2+3n-1}.$$

Solution.

a) Note that $\ln n > 1$ for $n \geq 3$, and therefore

$$\frac{1}{n^{3/2}\ln n} < \frac{1}{n^{3/2}}$$

for all $n \geq 3$. Therefore, since the series $\sum_{n=2}^{\infty} \frac{1}{n^{3/2}}$ converges as it is a generalized harmonic series with $p > 1$, then the series $\sum_{n=2}^{\infty} \frac{1}{n^{3/2}\ln n}$ converges as well by the first comparison test.

b) We will compare this series to the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$. We have

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{(n^2+1)n^2}{n^4+n-1} = 1 \neq 0;$$

therefore, since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges as a generalized harmonic series with $p > 1$, the series $\sum_{n=1}^{\infty} \frac{n^2+1}{n^4+n-1}$ is convergent by the second comparison test.

c) We will compare this series to the series $\sum_{n=1}^{\infty} \frac{1}{n}$. We have

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{(n+1)n}{n^2+3n-1} = 1 \neq 0;$$

therefore, since $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, being the harmonic series, the series $\sum_{n=1}^{\infty} \frac{n+1}{n^2+3n-1}$ also diverges by the second comparison test.

Example 7.4. Determine whether or not the following series are convergent:

$$\text{a) } \sum_{n=1}^{\infty} \frac{n+1}{n!}; \qquad \text{b) } \sum_{n=1}^{\infty} \frac{2^n}{n+3}; \qquad \text{c) } \sum_{n=1}^{\infty} \left(\frac{n}{2n+1}\right)^n.$$

Solution.

a) We will use the ratio test:

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+2)n!}{(n+1)!(n+1)} = \lim_{n\to\infty} \frac{n+2}{(n+1)^2} = 0.$$

As $0 < 1$, this series is convergent.

b) We will use the ratio test:

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{2^{n+1}(n+3)}{(n+4)2^n} = \lim_{n\to\infty} \frac{2(n+3)}{n+4} = 2.$$

As $2 > 1$, this series is divergent.

c) We will use the root test:

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \frac{n}{2n+1} = \frac{1}{2}.$$

As $0.5 < 1$, this series is convergent.

Determine whether or not the following series are convergent.

7.7. $\sum_{n=1}^{\infty} \frac{\ln n}{n}$

7.8. $\sum_{n=1}^{\infty} \frac{1}{\ln(n+1)}$

7.9. $\sum_{n=2}^{\infty} \frac{1}{\ln(2n)}$

7.10. $\sum_{n=1}^{\infty} \ln\left(2+\frac{2}{\sqrt{n}}\right)$

7.11. $\sum_{n=1}^{\infty} \frac{n}{2n-1}$

7.12. $\sum_{n=1}^{\infty} \frac{n-10}{100n+10}$

7.13. $\sum_{n=1}^{\infty} \frac{2}{n^3}$

7.14. $\sum_{n=1}^{\infty} \frac{n}{1+n^2}$

7.15. $\sum_{n=1}^{\infty} \frac{n}{(1+n)^3}$

7.16. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{(n+1)(n+2)}}$

7.17. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{(n+1)(n+2)(n+3)}}$

7.18. $\sum_{n=1}^{\infty} \frac{1}{n+n\sqrt[3]{n}}$

7.19. $\sum_{n=1}^{\infty} \frac{1}{2^n}\left(1+\frac{1}{n}\right)^{n^2}$

7.20. $\sum_{n=1}^{\infty} \left(\frac{n+1}{3n}\right)^n$

7.21. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{2n+n^2}}$

7.22. $\sum_{n=1}^{\infty} \frac{2n-1}{(\sqrt{2})^n}$

7.23. $\sum_{n=1}^{\infty} \frac{n\sqrt[3]{n}}{\sqrt[4]{n^7+3}}$

7.24. $\sum_{n=1}^{\infty} \frac{n^3}{4^n}$

7.25. $\sum_{n=1}^{\infty} \frac{4^n}{n3^n}$

7.26. $\sum_{n=1}^{\infty} \frac{1}{3^n\sqrt{n}}$

7.27. $\sum_{n=1}^{\infty} \frac{n}{(n+1)!}$

7.28. $\sum_{n=1}^{\infty} \frac{n+2}{n!4^n}$

7.29. $\sum_{n=1}^{\infty} \frac{2^n}{(n+5)!}$

7.30. $\sum_{n=1}^{\infty} \frac{n5^n}{(2n+1)!}$

7.31. $\sum_{n=1}^{\infty} \frac{n^n}{(n+1)!}$

7.32. $\sum_{n=1}^{\infty} \frac{n^n}{2^n(n+1)!}$

7.33. $\sum_{n=1}^{\infty} \frac{n^n}{3^n(n+1)!}$

7.34. $\sum_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^n$

7.35. $\sum_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^{n^2}$

7.36. $\sum_{n=1}^{\infty} \frac{1}{\ln^n(n+1)}$

7.37. $\sum_{n=1}^{\infty} \tan\frac{3+n}{n^2}$

7.38. $\sum_{n=1}^{\infty} \left(\tan^{-1}\left(\frac{1}{n}\right)\right)^n$

7.39. $\sum_{n=1}^{\infty} \sin\frac{\pi}{2^n}$

7.40. Consider the divergent series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$. What can be said about the convergence of i) $\sum_{n=1}^{\infty}(a_n+b_n)$; ii) $\sum_{n=1}^{\infty}a_nb_n$; iii) $\sum_{n=1}^{\infty}\frac{a_n}{b_n}$, assuming $b_n \neq 0$?

7.41. If $\sum_{n=1}^{\infty}a_n$ converges and $|b_n|<|a_n|$ for all n, is it true that the series $\sum_{n=1}^{\infty}b_n$ converges as well?

7.42. Show that if $\sum_{n=1}^{\infty}a_n$ converges and $a_n \geq 0$, then $\sum_{n=1}^{\infty}(e^{a_n}-1)$ also converges.

7.3 Alternating series

Consider the series $\sum_{n=1}^{\infty}a_n$, in which the terms a_n may be either positive or negative. The first step in investigating such series is to consider the convergence of $\sum_{n=1}^{\infty}|a_n|$.

Theorem (Sufficient condition for the convergence of a series)

If the series of absolute values $\sum_{n=1}^{\infty}|a_n|$ converges, then the series $\sum_{n=1}^{\infty}a_n$ converges as well.

Definition. If $\sum_{n=1}^{\infty}|a_n|$ converges, then the series $\sum_{n=1}^{\infty}a_n$ converges **absolutely**.

Note that, by definition, any convergent positive series converges absolutely.

Definition. If $\sum_{n=1}^{\infty}|a_n|$ does not converge, and yet the series $\sum_{n=1}^{\infty}a_n$ does converge, then it converges **conditionally**.

Definition. An **alternating series** is a series which can be written as $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$, where $a_n>0$, e.g.

$$a_1-a_2+a_3-a_4+...$$

Theorem (Leibniz's theorem)

If 1) $a_n>0$; 2) $\lim_{n\to\infty}a_n=0$; and 3) the terms a_n are non-increasing, then the alternating series $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ converges.

Theorem. If the series $\sum_{n=1}^{\infty}a_n$ converges absolutely, then it will remain absolutely convergent for any rearrangement of its terms, and its sum will not change.

Theorem. If the series $\sum_{n=1}^{\infty}a_n$ converges conditionally, then for any number A it is possible to rearrange the terms of the series so that its sum equals A. It is even possible to rearrange the terms so that the series becomes divergent!

Example 7.5. Determine whether or not the following series converge absolutely, conditionally, or diverge:

$$\text{a) } \sum_{n=1}^{\infty} \frac{(-1)^n}{n\sqrt{n}}; \qquad \text{b) } \sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n}; \qquad \text{c) } \sum_{n=1}^{\infty} \frac{(-1)^n n}{5n-2}.$$

Solution.

a) Consider the series of absolute values $\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}}$. This series converges, as it is a generalized harmonic series with $p = 1.5 > 1$. Therefore, the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n\sqrt{n}}$ converges absolutely.

b) Consider the series of absolute values $\sum_{n=2}^{\infty} \frac{1}{\ln n}$. Since $\ln n < n$ for $n \geq 2$, then we have

$$\frac{1}{\ln n} > \frac{1}{n}.$$

The harmonic series $\sum_{n=2}^{\infty} \frac{1}{n}$ diverges, and therefore the series $\sum_{n=2}^{\infty} \frac{1}{\ln n}$ also diverges by the first comparison test.

On the other hand, we have $\lim_{n\to\infty} \frac{1}{\ln n} = 0$ and, since $\ln x$ is an increasing function, $\frac{1}{\ln(n+1)} < \frac{1}{\ln n}$. Therefore, the terms of the series are non-increasing, and so by Leibnitz's theorem we conclude that the alternating series $\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n}$ converges conditionally.

c) For this case, it is enough to note that

$$\lim_{n\to\infty} \frac{n}{5n-2} = \frac{1}{5}.$$

Therefore, the terms of the series do not approach zero, and the series is divergent by the necessary condition of convergence.

Determine whether or not the following series converge absolutely, conditionally, or diverge.

7.43. $\sum_{n=1}^{\infty} \frac{\cos n}{n^5}$

7.44. $\sum_{n=1}^{\infty} \frac{\sin n}{3^n}$

7.45. $\sum_{n=1}^{\infty} \sin \frac{1}{n\sqrt{n}}$

7.46. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n+3}{n+1}$

7.47. $\sum_{n=1}^{\infty} \frac{(-1)^n}{2n+1}$

7.48. $\sum_{n=6}^{\infty} \frac{(-1)^{n+1}}{n^2-5n}$

7.49. $\sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{n^3-1}$

7.50. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n+1}{3n^2}$

7.51. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n\sqrt{n}}$

7.52. $\sum_{n=1}^{\infty} \frac{(-1)^n}{(n+1)4^{n-1}}$

7.53. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n!}$

7.54. $\sum_{n=1}^{\infty} \frac{(-1)^n n}{n!}$

7.55. $\sum_{n=1}^{\infty} \frac{(-1)^n}{2n+1}$

7.56. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^n}{n!}$

7.57. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{(n+1)!}{(n+1)^n}$

7.58. $\sum_{n=1}^{\infty} (-1)^{n+1} \left(\frac{n}{3n-1}\right)^n$

7.59. $\sum_{n=2}^{\infty} \frac{(-1)^n}{\sqrt{n}-1}$

7.60. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2}{5^n}$

7.61. $\sum_{n=1}^{\infty} (-1)^{n-1} \tan \frac{1}{\sqrt{n(n+1)}}$

7.62. Show that the series $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{2+(-1)^n}{n}$ is divergent. Why isn't Leibnitz's theorem applicable?

7.4 Power series

Definition. A **power series** is a function of the form

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots + a_n x^n + \ldots = \sum_{n=0}^{\infty} a_n x^n.$$

The numbers $a_0, a_1, \ldots$ are called the **coefficients** of the power series.

Power series, depending on the value of x, may be either convergent or divergent. However, such power series are *always* convergent for $x = 0$.

Definition. The set of values of x for which a power series is convergent is the **convergence set**.

Theorem. 1) If a power series is convergent for some value of x_1, where $x_1 \neq 0$, then it is *absolutely* convergent for any value of x such that $|x| < |x_1|$.
2) If a power series is divergent for some value of x_2, then it is divergent for any value of x such that $|x| > |x_2|$.

Therefore, there is some number R such that the power series converges absolutely for all $|x| < R$ and diverges for all $|x| > R$.

Definition. The **convergence interval** of a power series is the interval $(-R, R)$, such that the power series is convergent for $x \in (-R, R)$ and divergent for all $|x| > R$. R is called the **radius of convergence**.

Note that these theorems do not determine whether or not the series converges for $x = R$ and $x = -R$. It is therefore necessary to consider the endpoints of the convergence interval separately.

It is possible for R to equal 0 (in which case the power series is convergent only for $x = 0$), or ∞ (in which case the power series is convergent for all x).

Theorem. The radius of convergence can be found by using either of the following formulas:

$$R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| \qquad or \qquad R = \frac{1}{\lim\limits_{n\to\infty} \sqrt[n]{|a_n|}}.$$

Properties of power series

Theorem. The function $f(x) = \sum\limits_{n=0}^{\infty} a_n x^n$ is continuous on its convergence set.

Theorem. The function $f(x) = \sum\limits_{n=0}^{\infty} a_n x^n$ is differentiable on its convergence set:

$f'(x) = \sum\limits_{n=1}^{\infty} n a_n x^{n-1}$,
and the convergence interval of this series is the same as that of $f(x)$.

Power series are often written in a more general form—not in terms of x, but in terms of $x - x_0$:

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n.$$

The properties of such series are exactly the same as those considered above, except that the convergence interval will be $(x_0 - R, x_0 + R)$.

Example 7.6. Find the convergence set for the power series $1 + x + x^2 + \ldots$

Solution. First we will find the radius of convergence. Since $a_n = 1$, we have

$$R = \lim_{n\to\infty} \frac{1}{1} = 1.$$

Therefore, the series converges absolutely for all $|x| < 1$ and diverges for all $|x| > 1$. Now check the endpoints: for $x = 1$ the series $\sum\limits_{n=0}^{\infty} 1$ is obviously divergent because the necessary condition for convergence is not satisfied; for $x = -1$ the series $\sum\limits_{n=0}^{\infty} (-1)^n$ is divergent as well, and for the same reason.

Therefore, the series is convergent for $x \in (-1, 1)$.

Example 7.7. Find the convergence set for the power series

$$2x - \frac{(2x)^2}{2} + \frac{(2x)^3}{3} - \ldots + (-1)^{n+1} \frac{(2x)^n}{n} + \ldots$$

Solution. Here we have $a_n = (-1)^{n+1} \frac{2^n}{n}$, and so

$$R = \lim_{n\to\infty} \left| \frac{(-1)^{n+1} \frac{2^n}{n}}{(-1)^{n+2} \frac{2^{n+1}}{n+1}} \right| = \lim_{n\to\infty} \frac{n+1}{2n} = \frac{1}{2}.$$

This means that the power series will converge absolutely for all $|x| < 0.5$. At the left endpoint of the convergence interval ($x = -0.5$), the series will be written as

$$-1-\frac{1}{2}-\frac{1}{3}-\ldots=-\left(1+\frac{1}{2}+\frac{1}{3}+\ldots\right);$$

this is the harmonic series, and it is divergent. At the right endpoint $x=0.5$, however, the series will be written as

$$1-\frac{1}{2}+\frac{1}{3}-\ldots+(-1)^{n+1}\frac{1}{n}+\ldots;$$

this alternating series is conditionally convergent by Leibnitz's theorem. Therefore, the final answer is that the power series is convergent for $x\in(-0.5,0.5]$.

Example 7.8. Find the convergence set for the power series

$$\text{a) } \sum_{n=1}^{\infty}\frac{x^n}{n!} \qquad \text{and} \qquad \text{b) } \sum_{n=0}^{\infty}(nx)^n.$$

Solution. a) The radius of convergence of this series is

$$R=\lim_{n\to\infty}\frac{(n+1)!}{n!}=\lim_{n\to\infty}(n+1)=\infty;$$

therefore, this series is absolutely convergent for all x.

b) The radius of convergence of this series is

$$R=\lim_{n\to\infty}\frac{n^n}{(n+1)^{n+1}}=\lim_{n\to\infty}\frac{n^n}{(n+1)^n(n+1)}=$$

$$=\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n\frac{1}{n+1}=\lim_{n\to\infty}\frac{1}{e(n+1)}=0.$$ Therefore, this series is convergent only for $x=0$, and divergent for all other x.

Find the convergence set for the following power series.

7.63. $\sum_{n=0}^{\infty} n^3x^n$

7.64. $\sum_{n=0}^{\infty} 3^nx^n$

7.65. $\sum_{n=1}^{\infty} \frac{1}{n(n+3)}(x+8)^{n-1}$

7.66. $\sum_{n=0}^{\infty} \frac{1}{(2n-1)2^n}x^n$

7.67. $\sum_{n=0}^{\infty} \frac{(-1)^n}{n^3+1}x^{n+1}$

7.68. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n\sqrt{n}}x^{n+1}$

7.69. $\sum_{n=1}^{\infty} \frac{1}{n2^n}(x+4)^n$

7.70. $\sum_{n=0}^{\infty} \frac{(-1)^n}{(n^2+1)^2}(x-3)^n$

7.71. $\sum_{n=0}^{\infty} \frac{(x+1)^n}{(n+1)4^{n-1}}$

7.72. $\sum_{n=1}^{\infty} \frac{x^n}{2n+\sqrt[3]{n}}$

7.73. $\sum_{n=1}^{\infty} \frac{(x-2)^n}{\sqrt{n(n^2+1)}}$

7.74. $\sum_{n=0}^{\infty} \frac{1}{n\sqrt[3]{n}+1}(x+2)^n$

7.75. $\sum_{n=0}^{\infty} \frac{(-1)^n(x-5)^n}{\sqrt[4]{n}+1}$

7.76. $\sum_{n=0}^{\infty} \frac{n}{3n+1}\left(\frac{x}{4}\right)^n$

7.77. $\displaystyle\sum_{n=0}^{\infty} \frac{(-x)^n \sqrt{n+1}}{3^{n-1}}$

7.78. $\displaystyle\sum_{n=0}^{\infty} \frac{n^3}{(n+1)^2}(x+5)^n$

7.79. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n x^{n+1}}{\sqrt{n}3^{2n+1}}$

7.80. $\displaystyle\sum_{n=0}^{\infty} \frac{(-4)^{n+1}}{2n-1} x^{2(n-1)}$

7.81. $\displaystyle\sum_{n=1}^{\infty} \frac{(x+1)^n}{2^n n(n^2+1)!}$

7.82. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^n x^n$

Chapter 8.
TAYLOR AND MACLAURIN SERIES

Definition. Let $f(x)$ be a function defined on the open interval (a,b), and which can be differentiated $(n+1)$ times on (a,b). Then the equality

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \ldots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + R_{n+1}(x),$$

for any values of x and x_0 in (a,b) is called **Taylor's formula**. $R_{n+1}(x)$ is called the **remainder function**. The resulting function (without $R_{n+1}(x)$) is called **the Taylor expansion** of $f(x)$ with respect to x about the point $x = x_0$ of order n.

One of the most common forms of the remainder function is the **Lagrange** form:

$$R_{n+1}(x) = \frac{(x - x_0)^{n+1}}{(n+1)!} f^{(n+1)}\left(x_0 + \theta(x - x_0)\right),$$

where $0 < \theta < 1$.

Definition. If $\lim\limits_{n\to\infty} R_{n+1}(x) = 0$ for some x, then the infinite series

$$f(x) = f(x_0) + \sum_{n=1}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$

is called the **Taylor series** for $f(x)$. A **Maclaurin series** is a Taylor series with $x_0 = 0$.

Note that if $f(x)$ is a polynomial of degree n, then it will have at most only n non-zero derivatives; all other higher-order derivatives will be identically equal to zero.

Example 8.1. Write the power series $x^4 - 5x^3 + x^2 - 3x + 4$ in terms of $(x - 4)$.

Solution. First we will find the first 4 derivatives at $x = 4$:

$$f'(x) = 4x^3 - 15x^2 + 2x - 3; \qquad f'(4) = 21;$$

$$f''(x) = 12x^2 - 30x + 2; \qquad f''(4) = 74;$$

$$f'''(x) = 24x - 30; \qquad f'''(4) = 66;$$

$$f^{IV}(x) = 24.$$

Therefore, since $f(4) = -56$, the answer is

$$f(x) = -56+21(x-4)+\frac{74}{2!}(x-4)^2+\frac{66}{3!}(x-4)^3+\frac{24}{4!}(x-4)^4 =$$
$$= -56+21(x-4)+37(x-4)^2+11(x-4)^3+(x-4)^4.$$

8.1. Write the power series $3x^3+2x^2+x$ in terms of $x-1$.

8.2. Find the Taylor expansion of $f(x)=\frac{2}{x}$ about the point $x=1$ of order 3.

The following series are of extreme importance. All of them are Maclaurin series ($x_0=0$):

1. $e^x = 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots+\frac{x^n}{n!}+\ldots$
2. $\sin x = x-\frac{x^3}{3!}+\frac{x^5}{5!}-\ldots+(-1)^{n-1}\frac{x^{2n-1}}{(2n-1)!}+\ldots$
3. $\cos x = 1-\frac{x^2}{2!}+\frac{x^4}{4!}-\ldots+(-1)^n\frac{x^{2n}}{(2n)!}+\ldots$
4. If $-1<x<1$, then

$$(1+x)^m = 1+mx+\frac{m(m-1)}{2!}x^2+\ldots+\frac{m(m-1)(m-2)\ldots(m-n+1)}{n!}x^n+\ldots$$

In particular,

$$\frac{1}{1+x} = 1-x+x^2-x^3+x^4+\ldots+(-1)^n x^n+\ldots \quad (|x|<1)$$

5. If $-1<x\le 1$, then

$$\ln(1+x) = x-\frac{x^2}{2}+\frac{x^3}{3}-\ldots+(-1)^{n-1}\frac{x^n}{n}+\ldots$$

Using these formulas, it is possible to find the Taylor series for other functions without using Taylor's formula.

Example 8.2. Find the Taylor series for $f(x)=e^{2x}$ at $x_0=0$.

Solution. Denote $2x=t$, and note that $t\to 0$ as $x\to 0$. Using (1), we find

$$e^{2x} = e^t = 1+t+\frac{t^2}{2!}+\frac{t^3}{3!}+\ldots+\frac{t^n}{n!}+\ldots = 1+2x+\frac{4x^2}{2!}+\frac{8x^3}{3!}+\ldots+\frac{2^n}{n!}x^n+\ldots$$

Example 8.3. Find the Taylor series for $f(x)=\ln x$ at $x_0=1$.

Solution. $\ln x = \ln\big(1+(x-1)\big)$. Denote $x-1=t$, so that $t\to 0$ as $x\to 1$. Using (5), we find

$$\ln\big(1+(x-1)\big) = \ln(1+t) = t-\frac{t^2}{2}+\ldots+(-1)^{n-1}\frac{t^n}{n}+\ldots =$$
$$= (x-1)-\frac{(x-1)^2}{2}+\ldots+(-1)^{n-1}\frac{(x-1)^n}{n}+\ldots$$

This series is valid for $-1<x-1\le 1$, or for $0<x\le 2$.

Example 8.4. Find the Taylor series for $f(x) = \sqrt{x^3}$ at $x = 1$.

Solution. $\sqrt{x^3} = x^{3/2} = (1+(x-1))^{3/2}$. Denote $x-1=t$, so that $t \to 0$ as $x \to 1$. Using (4), we find

$$\sqrt{x^3} = (1+t)^{3/2} = 1 + \frac{3}{2}t + \frac{\frac{3}{2}\cdot\frac{1}{2}}{1\cdot 2}t^2 + \ldots + \frac{\frac{3}{2}\cdot\frac{1}{2}\cdot\ldots\cdot\left(\frac{3}{2}-n+1\right)}{n!}t^n + \ldots =$$

$$= 1 + \frac{3}{2}(x-1) + \frac{\frac{3}{2}\cdot\frac{1}{2}}{1\cdot 2}(x-1)^2 + \ldots + \frac{\frac{3}{2}\cdot\frac{1}{2}\cdot\ldots\cdot\left(\frac{3}{2}-n+1\right)}{n!}(x-1)^n + \ldots$$

Example 8.5. Find the Taylor series for $f(x) = \frac{1}{x}$ at $x = 3$.

Solution. $\frac{1}{x} = \frac{1}{3+(x-3)} = \frac{1}{3\left(1+\frac{x-3}{3}\right)}$. Denote $t = \frac{x-3}{3}$, so that $t \to 0$ as $x \to 3$. Using (4), we find

$$\frac{1}{x} = \frac{1}{3(1+t)} = \frac{1}{3}\left(1 - t + t^2 - t^3 + \ldots + (-1)^n t^n + \ldots\right) =$$

$$= \frac{1}{3}\left(1 - \frac{x-3}{3} + \left(\frac{x-3}{3}\right)^2 - \left(\frac{x-3}{3}\right)^3 + \ldots + (-1)^n\left(\frac{x-3}{3}\right)^n + \ldots\right) =$$

$$= \frac{1}{3} - \frac{x-3}{3^2} + \frac{(x-3)^2}{3^3} - \ldots + (-1)^n\frac{(x-3)^n}{3^{(n+1)}} + \ldots$$

This series is valid for $-1 < \frac{x-3}{3} < 1$, or $0 < x < 6$.

Example 8.6. Find the Taylor series for $f(x) = \sin\frac{\pi x}{4}$ at $x = 2$.

Solution. $\sin\frac{\pi x}{4} = \sin\left(\frac{\pi}{2} + \frac{\pi}{4}(x-2)\right) = \cos\left(\frac{\pi}{4}(x-2)\right)$. Denote $t = \frac{\pi}{4}(x-2)$, so that $t \to 0$ as $x \to 2$. Using (3), we find

$$\sin\frac{\pi x}{4} = \cos t = 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \ldots + (-1)^n\frac{t^{2n}}{(2n)!} + \ldots =$$

$$= 1 - \frac{\pi^2}{4^2\cdot 2!}(x-2)^2 + \frac{\pi^4}{4^4\cdot 4!}(x-2)^4 - \ldots + (-1)^n\frac{\pi^{2n}}{4^{2n}\cdot(2n)!}(x-2)^{2n} + \ldots$$

Example 8.7. Find the Taylor series for $f(x) = \begin{cases} \frac{e^x-1}{x}, & x \neq 0; \\ 1, & x = 0 \end{cases}$ at $x = 0$.

Solution.

$$f(x)=\frac{1+x+\frac{x^2}{2!}+\ldots+\frac{x^n}{n!}+\ldots-1}{x}=1+\frac{x}{2!}+\frac{x^2}{3!}+\ldots+\frac{x^{n-1}}{n!}+\ldots$$

Find the Taylor series for the following functions at the given point.

8.3. $f(x)=e^{-x^2}$, $x=0$.

8.4. $f(x)=\sin\frac{x}{2}$, $x=0$.

8.5. $f(x)=\cos^2 x$, $x=0$.

8.6. $f(x)=(x-\tan x)\cos x$, $x=0$.

8.7. $f(x)=\ln x$, $x=10$.

8.8. $f(x)=(x-1)\ln(x)$, $x=1$.

8.9. $f(x)=\sqrt{1+x^2}$, $x=0$.

8.10. $f(x)=\sqrt[3]{8-x^3}$, $x=0$.

8.11. $f(x)=\begin{cases}\frac{\sin x}{x}, & x\neq 0;\\ 1, & x=0,\end{cases}$ $x=0$.

8.12. $f(x) = \begin{cases}\frac{e^{x^3}-e^{-x^3}}{2x^3}, & x\neq 0;\\ 1, & x=0,\end{cases}$

$x=0$.

Example 8.8. Find the first five terms (i.e., up to the term x^4; a zero term is still a term!) in the Maclaurin expansion of the function $f(x)=\sin\left(e^x-1\right)$.

Solution. Instead of using Taylor's formula, we will use the expansions

$$e^x-1=x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\ldots$$

and

$$\sin y=y-\frac{1}{3!}y^3+\ldots$$

It follows from these, by putting

$$y=x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\ldots,$$

that

$$\sin\left(e^x-1\right)=\left(x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\ldots\right)-\frac{1}{3!}\left(x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\ldots\right)^3+\ldots$$

Consider separately the last expression, remembering that only terms involving x^4 or lower powers of x are of interest:

$$\left(x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\ldots\right)^3=\left(x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\ldots\right)^2\left(x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\ldots\right)=$$

$$=\left(x^2+x^3+\frac{7}{12}x^4+\ldots\right)\left(x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\ldots\right)=x^3+\frac{3}{2}x^4+\ldots$$

Therefore,

$$\sin\left(e^x-1\right)=\left(x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\dots\right)-\frac{1}{3!}\left(x^3+\frac{3}{2}x^4+\dots\right)+\dots=$$
$$=x+\frac{1}{2}x^2-\frac{5}{24}x^4+\dots$$

Note that, as always, it is possible to use Taylor's theorem directly. This would require finding the first four derivatives of $f(x)$ and calculating them at $x=0$. For the function considered in this example, using Taylor's theorem directly might be easier and faster; however, for more complex functions using Taylor's theorem is generally much more difficult and time-consuming.

Find the first five terms in the Maclaurin expansion of the following functions.

8.13. $f(x)=\ln(1+e^x)$.

8.14. $f(x)=\cos\left(\ln(1+x)\right)$

8.15. $f(x)=e^{\cos x}$.

8.16. $f(x)=\cos^n x$.

8.17. $f(x)=-\ln\cos x$.

8.18. $f(x)=(1+x)^x$.

Find the first non-zero term in the Maclaurin expansion of the following functions:

8.19. $x+\ln\left(\sqrt{1+x^2}-x\right)$.

8.20. $2-2\cos x-\sin^2 x$.

8.21. $\ln(1+2x)e^{2x}-2(x+x^2)$.

8.22. $\ln\left(1+x+x^2\right)+\ln\left(1-x+x^2\right)$.

8.23. $1-\cos(1-\cos x)$.

8.24. $e^{\tan 2x}-1-2x-2x^2$.

8.25. $\tan(\sin x)-\sin(\tan x)$.

8.26. $2\ln(2-\cos x)-\sin^2(x)$.

Chapter 9.
INDEFINITE INTEGRATION

Definition. If $F'(x) = f(x)$, then $F(x)$ is an **antiderivative** of $f(x)$.

Theorem. If $f(x)$ is continuous, then an antiderivative of $f(x)$ exists.

Note that if $F(x)$ is an antiderivative of $f(x)$, then any function $F(x)+C$, where C is a constant, is also an antiderivative; therefore, once an antiderivative of $f(x)$ has been found, infinitely many antiderivatives can also be found.

Theorem. If $F_1(x)$ and $F_2(x)$ are two antiderivatives of $f(x)$, then $F_1(x) - F_2(x)$ is a constant.

This theorem simply states that *all* antiderivatives of $f(x)$ can be written in this form $F(x)+C$, where $F(x)$ is any antiderivative of $f(x)$.

Definition. The set of all antiderivatives of $f(x)$ is called the **indefinite integral** of $f(x)$: $\int f(x)dx = F(x)+C$, where $F(x)$ is any antiderivative of $f(x)$ and C is an arbitrary constant.

Properties of the indefinite integral.

1. $\left(\int f(x)dx\right)' = f(x)$;

2. $d\left(\int f(x)dx\right) = f(x)dx$;

3. $\int dF(x) = F(x)+C$;

4. $\int kf(x)dx = k\int f(x)dx$, where k is any constant;

5. $\int \left(f(x) \pm g(x)\right) dx = \int f(x)dx \pm \int g(x)dx$.

Table of elementary integrals.

1. $\int x^a dx = \dfrac{x^{a+1}}{a+1} + C, \quad a \neq -1$;

2. $\int \dfrac{dx}{x} = \ln|x| + C$;

3. $\int \sin x dx = -\cos x + C, \quad \int \cos x dx = \sin x + C$;

4. $\displaystyle\int \frac{dx}{\cos^2 x} = \tan x + C, \quad \int \frac{dx}{\sin^2 x} = -\cot x + C;$

5. $\displaystyle\int a^x dx = \frac{a^x}{\ln a} + C,\ a>0,\ a\neq 1; \quad \int e^x dx = e^x + C;$

6. $\displaystyle\int \frac{dx}{1+x^2} = \tan^{-1} x + C, \quad \int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C;$

7. $\displaystyle\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| + C,\ a\neq 0;$

8. $\displaystyle\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + C, \quad \int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + C;$

9. $\displaystyle\int \frac{dx}{\sqrt{x^2\pm a^2}} = \ln\left|x+\sqrt{x^2\pm a^2}\right| + C,\ a\neq 0.$

9.1 Direct integration

Example 9.1. Find the following integrals using the table of elementary integrals:

$$\text{a)} \int \frac{dx}{x^4}; \quad \text{b)} \int \sqrt[3]{x}dx; \quad \text{c)} \int \frac{dx}{\sqrt{x}}.$$

Solution.

$$\text{a)} \int \frac{dx}{x^4} = \int x^{-4}dx = \frac{x^{-3}}{-3} + C = -\frac{1}{3x^3} + C;$$

$$\text{b)} \int \sqrt[3]{x}dx = \int x^{1/3}dx = \frac{x^{\frac{1}{3}+1}}{\frac{1}{3}+1} + C = \frac{3}{4}x^{4/3} + C;$$

$$\text{c)} \int \frac{dx}{\sqrt{x}} = \int x^{-1/2}dx = \frac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + C = 2\sqrt{x} + C.$$

Example 9.2. Find the following integrals using the table of elementary integrals

$$\text{a)} \int \frac{dx}{3^x}; \quad \text{b)} \int 2^{3x-1}dx; \quad \text{c)} \int \frac{dx}{9x^2-1}; \quad \text{d)} \int \frac{dx}{4x^2+25}; \quad \text{e)} \int \frac{dx}{\sqrt{4x^2+1}}.$$

Solution.

$$\text{a)} \int \frac{dx}{3^x} = \int \left(\frac{1}{3}\right)^x dx = \frac{\left(\frac{1}{3}\right)^x}{\ln\frac{1}{3}} + C = -\frac{1}{3^x \ln 3} + C;$$

$$\text{b)} \int 2^{3x-1}dx = \int \frac{8^x}{2}dx = \frac{1}{2}\frac{8^x}{\ln 8} + C = \frac{2^{3x}}{6\ln 2} + C;$$

$$\text{c)} \int \frac{dx}{9x^2-1} = \frac{1}{9}\int \frac{dx}{x^2-\frac{1}{9}} = \frac{1}{9}\frac{1}{2\left(\frac{1}{3}\right)}\ln\left|\frac{x-\frac{1}{3}}{x+\frac{1}{3}}\right| + C = \frac{1}{6}\ln\left|\frac{3x-1}{3x+1}\right| + C;$$

d) $\displaystyle\int \frac{dx}{4x^2+25} = \frac{1}{4}\int \frac{dx}{x^2+\frac{25}{4}} = \frac{1}{10}\tan^{-1}\frac{2x}{5}+C;$

e) $\displaystyle\int \frac{dx}{\sqrt{4x^2+1}} = \frac{1}{2}\int \frac{dx}{\sqrt{x^2+\frac{1}{4}}} = \frac{1}{2}\ln\left|x+\sqrt{x^2+\tfrac{1}{4}}\right|+C.$

Find the following integrals using the table of elementary integrals.

9.1. $\int x^4 dx.$

9.2. $\int \frac{dx}{2\sqrt{x}}.$

9.3. $\int (5x^3+6x^2-3x+1)dx.$

9.4. $\int 3x^{-0.35}dx.$

9.5. $\int \left(\frac{x}{2}+1\right)^3 dx.$

9.6. $\int \frac{-2dx}{x^2}.$

9.7. $\int 5^x dx.$

9.8. $\int \frac{(x^2+2)^2}{\sqrt[3]{x}}dx.$

9.9. $\int \frac{dx}{2x^2-3}.$

9.10. $\int \left(\sqrt{x}+\frac{1}{\sqrt[3]{x}}\right)^2 dx.$

9.11. $\int \frac{2x+4}{x}dx.$

9.12. $\int \frac{x^2}{x^2+4}dx.$

9.13. $\int \frac{x^2-x}{3x}dx.$

9.14. $\int \frac{x^5-x+1}{x^2+1}dx.$

9.15. $\int \frac{x^2-16}{\sqrt{x}+2}dx.$

9.16. $\int \sin^2\left(\frac{x}{2}\right) dx.$

9.17. $\int \cos^2\left(\frac{x}{2}\right) dx.$

9.18. $\int \frac{\sin 2x}{\cos x}dx.$

9.19. $\int \tan^2 x dx.$

9.20. $\int \cot^2 x dx.$

9.21. $\int \sec^2 x dx.$

9.22. $\int \csc^2 x dx.$

9.23. $\int \frac{1+3^x}{5^x}dx.$

9.24. $\int \left(\sin^{-1}x+\cos^{-1}x\right) dx.$

9.25. $\int \left(\tan^{-1}x+\cot^{-1}x\right) dx.$

9.2 Integration by substitution

Suppose that $\int f(x)dx$ cannot be written in the form of elementary integrals. In some cases the integral can be found by substitution. Substitution involves using some function of the form $x = x(t)$ in order to transform the integral $\int f(x)dx$ into an integral of the form $\int g(t)dt$, which can be subsequently found.

Example 9.3. Find the following integrals:

$$\text{a) } \int \cos(3x)dx; \quad \text{b) } \int \frac{xdx}{1+x^2}; \quad \text{c) } \int \sqrt{\sin x}\cos x dx;$$

$$\text{d) } \int \tan x dx; \quad \text{e) } \int (2x^3+1)^4 x^2 dx.$$

Solution.

a) Let $t = 3x$, then $dt = 3dx$ and $dx = \frac{1}{3}dt$. Therefore,

$$\int \cos(3x)dx = \int \cos t \frac{dt}{3} = \frac{1}{3}\sin t + C = \frac{1}{3}\sin(3x) + C.$$

b) Let $t = 1+x^2$, then $dt = 2xdx$ and $xdx = \frac{1}{2}dt$. Therefore,

$$\int \frac{xdx}{1+x^2} = \frac{1}{2}\int \frac{dt}{t} = \frac{1}{2}\ln|t| + C = \frac{1}{2}\ln(1+x^2) + C.$$

c) Let $t = \sin x$, then $dt = \cos x dx$. Therefore,

$$\int \sqrt{\sin x}\cos x dx = \int \sqrt{t}dt = \frac{2}{3}t^{3/2} + C = \frac{2}{3}(\sin x)^{3/2} + C.$$

d) Rewrite $\tan x$ as $\frac{\sin x}{\cos x}$. Let $t = \cos x$, then $dt = -\sin x dx$ and $\sin x dx = -dt$. Therefore,

$$\int \tan x dx = \int \frac{\sin x}{\cos x}dx = -\int \frac{dt}{t} = -\ln|t| + C = -\ln|\cos x| + C.$$

e) Let $t = 2x^3+1$, then $dt = 6x^2dx$ and $x^2dx = \frac{1}{6}dt$. Therefore,

$$\int (2x^3+1)^4 x^2 dx = \frac{1}{6}\int t^4 dt = \frac{1}{6}\frac{t^5}{5} + C = \frac{(2x^3+1)^5}{30} + C.$$

Find the following integrals by substitution.

9.26. $\int \frac{e^{1-\sqrt{x}}dx}{\sqrt{x}}$.

9.27. $\int \frac{xdx}{\sqrt{2x+1}}$.

9.28. $\int (2x+1)e^{x^2+x}dx$.

9.29. $\int \frac{(2x+3)dx}{x^2+3x+10}$.

9.30. $\int \frac{(3x^2-2x-5)dx}{x^3-x^2-5x+7}$.

9.31. $\int \frac{x^3dx}{\sqrt{x-1}}$.

9.32. $\int \frac{xdx}{2x+3}$.

9.33. $\int \frac{4x+3}{(x-2)^2}dx$.

9.34. $\int \cos^2 x \sin^3 x dx$.

9.35. $\int \sec^2 x dx$.

9.36. $\int 2^{\cos x}\sin x dx$.

9.37. $\int \frac{dx}{16-9x^2}$.

9.38. $\displaystyle\int \frac{dx}{x\sqrt{x+1}}$.

9.39. $\displaystyle\int \frac{xdx}{\sin^2(x^2-1)}$.

9.40. $\displaystyle\int \frac{dx}{\sqrt{\tan x}\cos^2 x}$.

9.41. $\displaystyle\int \frac{3-\tan x}{\cos^2 x}dx$.

9.42. $\displaystyle\int \frac{dx}{\sqrt{1-x^2}(\sin^{-1} x)^3}$.

9.43. $\displaystyle\int \frac{e^x dx}{e^{2x}+4}$.

9.44. $\displaystyle\int \frac{x^3 dx}{\sqrt{1-x^8}}$.

9.45. $\displaystyle\int \frac{x^3 dx}{\sqrt{9-x^2}}$.

9.46. $\displaystyle\int \frac{(2x-4)dx}{x^2-4x+10}$.

9.47. $\displaystyle\int \frac{\sin\left(\frac{1}{x}+5\right)}{x^2}dx$.

9.48. $\displaystyle\int \frac{dx}{x\ln^6 x}$.

9.49. $\displaystyle\int \sin(4x)\cos^4(2x)dx$.

9.50. $\displaystyle\int x\cdot 3^{5x^2+4}dx$.

9.51. $\displaystyle\int \frac{dx}{\sqrt{(1-25x^2)\cos^{-1}5x}}$.

9.52. $\displaystyle\int \frac{(\tan^{-1} x)^2}{1+x^2}dx$.

9.53. $\displaystyle\int \frac{3x-1}{x^2+9}dx$.

9.54. $\displaystyle\int \frac{x+1}{x\sqrt{x-2}}dx$.

9.55. $\displaystyle\int \frac{2x-\sqrt{\sin^{-1} x}}{\sqrt{1-x^2}}dx$.

9.56. $\displaystyle\int \frac{3^{\tan x}dx}{\cos^2 x}$.

9.57. $\displaystyle\int \frac{dx}{\sqrt{x}+\sqrt[3]{x}}$.

9.58. $\displaystyle\int \frac{\sqrt{x}dx}{\sqrt{x}+2}$.

9.59. $\displaystyle\int \frac{\sqrt{x}dx}{\sqrt[4]{x}+1}$.

9.3 Integration by parts

Consider the functions $u(x)$ and $v(x)$. Integration of the equality $d(uv) = udv + vdu$ gives

$$\int d(uv) = \int udv + \int vdu,$$

so

$$\int udv = uv - \int vdu.$$

This formula (**integration by parts**) reduces finding $\int udv$ to finding $\int vdu$, which may be simpler to find.

Example 9.4. Find $\displaystyle\int \frac{x^2}{(1+x^2)^2}dx$.

Solution.

$$\int \frac{x^2}{(1+x^2)^2}dx = \left\{ \begin{array}{ll} u = x, & du = dx \\ dv = \dfrac{xdx}{(1+x^2)^2}, & v = -\dfrac{1}{2(1+x^2)} \end{array} \right\} =$$

$$= -\frac{x}{2(1+x^2)} + \frac{1}{2}\int \frac{dx}{1+x^2} = -\frac{x}{2(1+x^2)} + \frac{1}{2}\tan^{-1}x + C.$$

While integration by parts is a general method that can be used to find integrals, three major cases are generally of interest:

I. Integrals of the type $\int x^n \sin ax dx$, $\int x^n \cos ax dx$, $\int x^n e^{ax} dx$.

In this case, choose $u(x) = x^n$ and integrate by parts. It will be necessary to use integration by parts n times.

Example 9.5. Find $\int x\cos(2x)dx$.

Solution.

$$\int x\cos(2x)dx = \left\{ \begin{array}{ll} u = x, & du = dx \\ dv = \cos(2x)dx, & v = \dfrac{1}{2}\sin(2x) \end{array} \right\} =$$

$$= \frac{x}{2}\sin(2x) - \frac{1}{2}\int \sin(2x)dx = \frac{x}{2}\sin(2x) + \frac{1}{4}\cos(2x) + C.$$

II. Integrals containing logarithmic or inverse trigonometric functions.

In this case, choose $u(x)$ to be the logarithmic or inverse trigonometric function.

Example 9.6. Find the following integrals:

$$\text{a)} \int \frac{\ln x}{x^2}dx; \quad \text{b)} \int \tan^{-1}x dx.$$

Solution.

$$\text{a)} \int \frac{\ln x}{x^2}dx = \left\{ \begin{array}{ll} u = \ln x, & du = \dfrac{dx}{x} \\ dv = \dfrac{dx}{x^2}, & v = -\dfrac{1}{x} \end{array} \right\} = -\frac{\ln x}{x} + \int \frac{dx}{x^2} = -\frac{\ln x}{x} - \frac{1}{x} + C.$$

$$\text{b)} \int \tan^{-1}x dx = \left\{ \begin{array}{ll} u = \tan^{-1}x, & du = \dfrac{dx}{1+x^2} \\ dv = dx, & v = x \end{array} \right\} =$$

$$= x\tan^{-1}x - \int \frac{xdx}{1+x^2} = x\tan^{-1}x - \frac{1}{2}\ln(1+x^2) + C.$$

III. Integrals of the type $\int e^{ax}\sin bx dx$ and $\int e^{ax}\cos bx dx$.

In this case, it is necessary to integrate twice by parts, each time choosing $u(x)$ to be either the exponential or trigonometric function.

Example 9.7. Find $\int e^{2x} \sin x dx$.

Solution.

$$\int e^{2x} \sin x dx = \left\{ \begin{array}{ll} u = \sin x, & du = \cos x dx \\ dv = e^{2x} dx, & v = \frac{1}{2} e^{2x} \end{array} \right\} = \frac{1}{2} e^{2x} \sin x - \frac{1}{2} \int e^{2x} \cos x dx =$$

$$= \left\{ \begin{array}{ll} u = \cos x, & du = -\sin x dx \\ dv = e^{2x} dx, & v = \frac{1}{2} e^{2x} \end{array} \right\} = \frac{1}{2} e^{2x} \sin x - \frac{1}{2} \left(\frac{1}{2} e^{2x} \cos x + \frac{1}{2} \int e^{2x} \sin x dx \right) =$$

$$= \frac{1}{2} e^{2x} \sin x - \frac{1}{4} e^{2x} \cos x - \frac{1}{4} \int e^{2x} \sin x dx.$$

Moving $\frac{1}{4} \int e^{2x} \sin x dx$ to the left side of the equation, we have

$$\frac{5}{4} \int e^{2x} \sin x dx = \frac{1}{2} e^{2x} \sin x - \frac{1}{4} e^{2x} \cos x + C,$$

so

$$\int e^{2x} \sin x dx = \frac{e^{2x}}{5} (2 \sin x - \cos x) + C.$$

Find the following integrals using integration by parts.

9.60. $\int x \sin x dx$.

9.61. $\int x^2 \cos x dx$.

9.62. $\int x^2 \sin(2x) dx$.

9.63. $\int x e^{-x} dx$.

9.64. $\int x^2 e^x dx$.

9.65. $\int x^2 e^{-3x} dx$.

9.66. $\int \frac{x dx}{\sin^2 x}$.

9.67. $\int \frac{\ln x}{x^3} dx$.

9.68. $\int \ln x dx$.

9.69. $\int \cos^{-1} 3x dx$.

9.70. $\int \sqrt{1 - x^2} dx$.

9.71. $\int \sqrt{x^2 + 4} dx$.

9.72. $\int x^2 \ln(2x) dx$.

9.73. $\int (x+1) \ln(x+1) dx$.

9.74. $\int \frac{dx}{(x^2+4)^2}$.

9.75. $\int x \sqrt{1 - x^4} dx$.

9.76. $\int (x^2 - x) 3^{-x} dx$.

9.77. $\int \left(\tan^{-1} x - \cot^{-1} x \right) dx$.

9.78. $\int \ln(x^2+x+1)dx.$ **9.79.** $\int \tan^{-1}(x^2)dx.$

9.4 Integration of rational functions

Definition. A **rational function** is a function of the form $f(x) = \frac{P(x)}{Q(x)}$, where $P(x)$ and $Q(x)$ are polynomials.

Definition. A **proper rational function** is a rational function in which the degree of $P(x)$ is strictly less than the degree of $Q(x)$.

Improper rational functions (i.e. those in which the degree of $P(x)$ is greater than or equal to the degree of $Q(x)$) can always be rewritten as the sum of a polynomial and a proper rational function. Therefore, the problem of integrating of rational functions is really the problem of integrating *proper* rational functions.

The integrals of proper rational functions are found by **partial fraction expansion** of the integrand into simple fractions.

There are 4 types of simple fractions:

1. Fractions of the type $\frac{A}{x-a}$.

   The integrals of such fractions are easily found:

   $\int \frac{Adx}{x-a} = A\ln|x-a| + C.$

2. Fractions of the type $\frac{A}{(x-a)^n}$, where n is a whole number greater than 1.

   These fractions are also easily integrated:

   $$\int \frac{Adx}{(x-a)^n} = A\int (x-a)^{-n}dx = A\frac{(x-a)^{1-n}}{1-n} + C = \frac{A}{(1-n)(x-a)^{n-1}} + C.$$

3. Fractions of the type $\frac{Ax+B}{x^2+px+q}$, where $p^2-4q<0$.

   The integrals of such fractions are found by completing the square in the denominator and subsequent substitution. An example of finding the integral of a fraction of this type will be given in Example 9.8.

4. Fractions of the type $\frac{Ax+B}{(x^2+px+q)^n}$, where $p^2-4q<0$ and n is a whole number greater than 1.

   Integration of this type of fraction will not be considered in this course.

Example 9.8. Find a) $\int \frac{3dx}{2x+4}$; b) $\int \frac{3dx}{(2x+4)^3}$; c) $\int \frac{3x+2}{x^2+2x+2}dx$.

Solution.

a) $\int \frac{3dx}{2x+4} = \frac{3}{2}\int \frac{dx}{x+2} = \frac{3}{2}\ln|x+2|+C.$

b) $\int \frac{3dx}{(2x+4)^3} = \frac{3}{8}\int \frac{dx}{(x+2)^3} = -\frac{3}{16(x+2)^2}+C.$

c) First it is necessary to complete the square in the denominator:

$$\int \frac{3x+2}{x^2+2x+2}dx = \int \frac{3x+2}{(x+1)^2+1}dx.$$

Let $t = x+1$, so we will have

$$\int \frac{3x+2}{(x+1)^2+1}dx = \int \frac{3(t-1)+2}{t^2+1}dt = \int \frac{3tdt}{t^2+1} - \int \frac{dt}{t^2+1}.$$

We will use another substitution, $u = t^2+1$, to find the first integral; the second can be found immediately:

$$\int \frac{3tdt}{t^2+1} = \frac{3}{2}\int \frac{du}{u} = \frac{3}{2}\ln|u|+C = \frac{3}{2}\ln(t^2+1)+C;$$

$$\int \frac{dt}{t^2+1} = \tan^{-1}t+C.$$

Therefore, we find

$$\int \frac{3x+2}{(x+1)^2+1}dx = \frac{3}{2}\ln(t^2+1) - \tan^{-1}t+C =$$

$$= \frac{3}{2}\ln(x^2+2x+2) - \tan^{-1}(x+1)+C.$$

Expansion of proper rational functions in partial fractions is achieved by first factoring the denominator and then writing the type of partial fraction (with unknown coefficients in the numerator) that corresponds to each term in the denominator:

1. if the denominator contains $(x-a)$, then the partial fraction expansion will contain $\frac{A}{x-a}$;
2. if the denominator contains $(x-a)^2$, then the partial fraction expansion will contain $\frac{A}{(x-a)^2}+\frac{B}{x-a}$;
3. if the denominator contains $(x-a)^n$, then the partial fraction expansion will contain $\frac{A}{(x-a)^n}+\frac{B}{(x-a)^{n-1}}+\ldots+\frac{Z}{(x-a)}$;
4. if the denominator contains (x^2+px+q), where $p^2-4q<0$, then the partial fraction expansion will contain $\frac{Ax+B}{x^2+px+q}$.

The unknown coefficients (A, B, etc.) are then found by one of two ways: by inserting concrete values of x, or by using the method of undetermined coefficients.

Example 9.9. Find $\int \frac{dx}{x^2-4x+3}$.

Solution. The denominator of this rational function can factored and rewritten as $x^2-4x+3=(x-1)(x-3)$. The expansion in partial fractions will have the form

$$\frac{1}{(x-1)(x-3)}=\frac{A}{x-1}+\frac{B}{x-3}.$$

We want this transformation to be identical, i.e. the values of the left and right sides of this expression should be the same for all x. If we bring the terms in the right side to a common denominator, we will find

$$\frac{1}{(x-1)(x-3)}=\frac{A(x-3)+B(x-1)}{(x-1)(x-3)}.$$

This can be identically true only if $A(x-3)+B(x-1)=1$ for all x. In particular, if $x=3$, then we find $B=1/2$; and if $x=1$, then we find $A=-1/2$. Therefore,

$$\frac{1}{(x-1)(x-3)}=-\frac{1}{2}\frac{1}{x-1}+\frac{1}{2}\frac{1}{x-3}.$$

The validity of this equality can be checked by simplifying.

Alternatively, we could use the method of undetermined coefficients to find A and B. According to this method, we can rewrite the condition $A(x-3)+B(x-1)=1$ as

$$(A+B)x+(-3A-B)=1.$$

The method of undetermined coefficients consists of equating the coefficients for various powers of x on the left and right sides of this equation. The coefficient of x on the left is $(A+B)$, while on the right it is 0; therefore $A+B=0$. Equating the free terms on the left and right sides gives $-3A-B=1$. Solving these two equations, we find $A=-1/2$ and $B=1/2$, as before.

It is now possible to find the integral:

$$\int \frac{dx}{x^2-4x+3}=-\frac{1}{2}\int \frac{dx}{x-1}+\frac{1}{2}\int \frac{dx}{x-3}=-\frac{1}{2}\ln|x-1|+\frac{1}{2}\ln|x-3|+C,$$

which can also be written more concisely as

$$\int \frac{dx}{x^2-4x+3}=\frac{1}{2}\ln\left|\frac{x-3}{x-1}\right|+C.$$

Example 9.10. Find

$$\text{a)} \int \frac{4x+5}{x^2-4x+3}dx; \quad \text{b)} \int \frac{x^2+1}{(x-1)(x-2)(x-3)}dx; \quad \text{c)} \int \frac{xdx}{(x+1)^2(x+2)}.$$

Solution.

a) First it is necessary to factor the denominator: $x^2-4x+3=(x-1)(x-3)$. The next step is to rewrite the integrand in partial fractions:

$$\frac{4x+5}{(x-1)(x-3)} = \frac{A}{x-1} + \frac{B}{x-3} = \frac{A(x-3)+B(x-1)}{(x-1)(x-3)}.$$

Therefore, we have $A(x-3)+B(x-1) = 4x+5$ for all x. For $x=3$ we find $B=17/2$, while for $x=1$ we find $A=-9/2$. Therefore,

$$\frac{4x+5}{(x-1)(x-3)} = \frac{-9/2}{x-1} + \frac{17/2}{x-3}.$$

It is now possible to find the integral:

$$\int \frac{4x+5}{x^2-4x+3}dx = -\frac{9}{2}\int \frac{dx}{x-1} + \frac{17}{2}\int \frac{dx}{x-3} = -\frac{9}{2}\ln|x-1| + \frac{17}{2}\ln|x-3| + C.$$

b) Again, it is first necessary to rewrite the integrand in partial fractions:

$$\frac{x^2+1}{(x-1)(x-2)(x-3)} = \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x-3} =$$
$$\frac{A(x-2)(x-3)+B(x-1)(x-3)+C(x-1)(x-2)}{(x-1)(x-2)(x-3)}.$$

We must find the constants A, B and C such that the numerators of the first and last fractions are the same. For $x=1$ we find $2A=2$ and therefore $A=1$; for $x=2$ we find $-B=5$ and therefore $B=-5$; finally, for $x=3$ we find $2C=10$ and therefore $C=5$:

$$\frac{x^2+1}{(x-1)(x-2)(x-3)} = \frac{1}{x-1} - \frac{5}{x-2} + \frac{5}{x-3}.$$

The integral is therefore equal to

$$\int \frac{x^2+1}{(x-1)(x-2)(x-3)}dx = \ln|x-1| - 5\ln|x-2| + 5\ln|x-3| + C.$$

c) The integrand can be rewritten in partial fractions as follows:

$$\frac{x}{(x+1)^2(x+2)} = \frac{A}{(x+1)^2} + \frac{B}{x+1} + \frac{C}{x+2} = \frac{A(x+2)+B(x+1)(x+2)+C(x+1)^2}{(x+1)^2(x+2)}.$$

We demand that $A(x+2)+B(x+1)(x+2)+C(x+1)^2 = x$ for all x. In particular, for $x=-2$ we find $C=-2$; for $x=-1$ we find $A=-1$. We can use other value of x for determining B; for instance for $x=0$ we have $2A+2B+C=0$, and therefore $B=2$. Therefore,

$$\frac{x}{(x+1)^2(x+2)} = \frac{-1}{(x+1)^2} + \frac{2}{x+1} + \frac{-2}{x+2}.$$

This equality can be checked by simplification. The initial integral is now easy to find:

$$\int \frac{xdx}{(x+1)^2(x+2)} = -\int \frac{dx}{(x+1)^2} + 2\int \frac{dx}{x+1} - 2\int \frac{dx}{x+2} =$$

$$= \frac{1}{x+1} + 2\ln|x+1| - 2\ln|x+2| + C = \frac{1}{x+1} + 2\ln\left|\frac{x+1}{x+2}\right| + C.$$

Example 9.11. Find $\int \frac{x^2+10x-10}{x(x^2+2x+2)}dx$.

Solution. The denominator of the integrand of this integral contains a quadratic polynomial with a negative discriminant. Therefore, the integrand will have the following representation in partial fractions:

$$\frac{x^2+10x-10}{x(x^2+2x+2)} = \frac{A}{x} + \frac{Bx+C}{x^2+2x+2}.$$

After finding the common denominator on the right side of this equation, we will come to the condition

$$x^2+10x-10 = A(x^2+2x+2) + (Bx+C)x$$

for all x. Using the method of undetermined coefficients, this condition can be rewritten as

$$(A+B)x^2 + (2A+C)x + 2A = x^2+10x-10,$$

leading to the system of equations

$$\begin{cases} A+B = 1; \\ 2A+C = 10; \\ 2A = -10. \end{cases}$$

The solution is $A=-5$, $B=6$, $C=20$. Therefore,

$$\frac{x^2+10x+10}{x(x^2+2x+2)} = -\frac{5}{x} + \frac{6x+20}{x^2+2x+2}.$$

The integral of the first item is

$$\int \frac{5dx}{x} = 5\ln|x| + C;$$

the second is

$$\int \frac{(6x+20)dx}{x^2+2x+2} = \int \frac{(6x+20)dx}{(x+1)^2+1} = \left\{ \begin{matrix} t = x+1 \\ dx = dt \end{matrix} \right\} = \int \frac{(6t+14)dt}{t^2+1} =$$

$$= 3\int \frac{d(t^2+1)}{t^2+1} + 14\int \frac{dt}{t^2+1} =$$

$$= 3\ln(t^2+1) + 14\tan^{-1}t + C = 3\ln(x^2+2x+2) + 14\tan^{-1}(x+1) + C.$$

Therefore, the final answer is

$$\int \frac{x^2+10x-10}{x(x^2+2x+2)}dx = -5\ln|x| + 3\ln(x^2+2x+2) + 14\tan^{-1}(x+1) + C.$$

In order to integrate an improper rational function, it is first necessary to reduce the integrand to the sum of a polynomial and a proper rational function by long division.

Example 9.12. Find $\int \frac{x^3+x^2+3x+4}{x^2+x-6}dx$.

Solution. The first thing that should be noticed in this example is that the integrand is not a proper rational function (the degree of the numerator is greater than the degree of the denominator). Therefore, it is first necessary to reduce the integrand by long division; we will find

$$\frac{x^3+x^2+3x+4}{x^2+x-6} = x + \frac{9x+4}{x^2+x-6}.$$

Therefore, we now have

$$\int \frac{x^3+x^2+3x+4}{x^2+x-6}dx = \frac{1}{2}x^2 + \int \frac{9x+4}{x^2+x-6}dx.$$

All that is left is to find the last integral, which we do by simplifying the integrand:

$$\frac{9x+4}{x^2+x-6} = \frac{9x+4}{(x+3)(x-2)} = \frac{A}{x+3} + \frac{B}{x-2}.$$

Equating the numerators we find $A(x-2)+B(x+3) = 9x+4$ for all x; inserting $x=2$ gives $B = 22/5$, and inserting $x=-3$ gives $A = 23/5$. Therefore,

$$\int \frac{x^3+x^2+3x+4}{x^2+x-6}dx = \frac{1}{2}x^2 + \frac{23}{5}\int \frac{dx}{x+3} + \frac{22}{5}\int \frac{dx}{x-2} =$$
$$= \frac{1}{2}x^2 + \frac{23}{5}\ln|x+3| + \frac{22}{5}\ln|x-2| + C.$$

Find the following integrals:

9.80. $\int \frac{dx}{x^2-9x+20}$

9.81. $\int \frac{2x+5}{x^2-9x+20}dx$

9.82. $\int \frac{x^2-x+1}{x^2-9x+20}dx$

9.83. $\int \frac{x^3dx}{x^2-9x+20}$

9.84. $\int \frac{3x^3+11x^2+x+3}{x+3}dx$

9.85. $\int \frac{12x^2+3}{2x^2+x-1}dx$

9.86. $\int \frac{3x+10}{(x-1)(x+2)(x+4)}dx$

9.87. $\int \frac{3x-2}{(x+3)^2}dx$

9.88. $\int \frac{x-1}{x+2}dx$

9.89. $\int \frac{dx}{(x-1)^2(x+1)}$

9.90. $\int \frac{dx}{(x+2)^2(x+3)^2}$

9.91. $\int \frac{x^2+10x+24}{(x-2)^2(x+1)}dx$

9.92. $\int \frac{x+2}{x-1} dx$

9.93. $\int \frac{2x dx}{(x^2+1)(x-1)}$

9.94. $\int \frac{3dx}{x^2+2x+5}$

9.95. $\int \frac{x^2+4}{(x^2+2x+2)(x+2)} dx$

9.96. $\int \frac{x+15}{(x^2+4)(x^2+9)} dx$

9.97. $\int \frac{4x^2+2}{(x^2+1)(x+1)^2} dx$

Chapter 10.
INTRODUCTION TO DIFFERENTIAL EQUATIONS

10.1 Basic concepts

Definition. A **first-order differential equation** is an equation that establishes a relationship between the independent variable x, the unknown function $y = f(x)$ and its first derivative y':

$$F(x, y, y') = 0.$$

Such differential equations are called **ordinary** because the unknown function $y = f(x)$ is a function of only one variable.

Definition. The function $y(x)$ is a **solution** of a differential equation if it satisfies the equation for any value of x.

Definition. The graph of the solution of a differential equation is an **integral curve**.

Example 10.1. Consider the differential equation

$$y' = 2y.$$

One of the solutions to this differential equation is, for example, $y = 3e^{2x}$; indeed,

$$y' = \left(3e^{2x}\right)' = 6e^{2x} = 2\left(3e^{2x}\right) = 2y.$$

Note that there are infinitely many solutions to this differential equation; it can be shown that all of these solutions have the form $y = Ce^{2x}$, where C is an arbitrary constant.

10.1. Verify that the function $y = 5e^{4x} + e^{3x}$ is a solution to the differential equation $y' - 4y = -e^{3x}$.

10.2. Verify that the function $y = \tan x$ is a solution to the differential equation $y' - y^2 = 1$.

10.3. Verify that the function $y = e^x - e^{-x}$ is a solution to the differential equation $(y')^2 = 4 + y^2$.

10.4. Verify that the function $y = x^7$ is a solution to the differential equation $x^2 y'' = 6xy'$.

10.5. The differential equation $x^2 y'' + 35y = 11xy'$ admits a solution of the form $y = x^a$, where $a \neq 0$. What are the possible values of a?

Definition. An **initial condition** is an additional equation that gives the value of the unknown function $y = f(x)$ at some point:

$$y(x)\big|_{x=x_0} = y_0, \qquad \text{or} \qquad y(x_0) = y_0.$$

Definition. The **general solution** of a differential equation is a function of the form $y = g(x,C)$ that satisfies two conditions:

1) $g(x,C)$ is a solution to the differential equation for any value of C;
2) for any given initial value condition $y(x_0) = y_0$ it is possible to find a particular value of the arbitrary constant C_0 such that the function $g(x,C_0)$ satisfies the initial condition, i.e. $g(x_0,C_0) = y_0$.

Example 10.2. In Example 10.1 the general solution is $y = Ce^{2x}$.

Definition. An **initial value problem** is a system consisting of a differential equation and an initial condition.

Definition. The **particular** or **definite solution** of a differential equation is the one solution that satisfies a given initial condition.

Theorem (Existence and uniqueness of the solution to an initial value problem)

Consider the differential equation $y' = f(x,y)$. If the function $f(x,y)$ and its derivative $\dfrac{\partial f}{\partial y}$ is continuous in some region D on the coordinate plane that contains the point (x_0,y_0), then there exists exactly one solution $y(x)$ that satisfies both the differential equation and the initial condition $y(x_0) = y_0$.

The notation $\frac{\partial f}{\partial y}$ is called the **partial derivative** of $f(x,y)$ with respect to y. It can be found by simply differentiating $f(x,y)$ with respect to y and considering x to be a fixed constant. For instance, if $f(x,y) = x^2y^3$, then $\frac{\partial f}{\partial y} = 3x^2y^2$.

Example 10.3. Consider the initial value problem

$$\begin{cases} y' = 2y, \\ y(0) = 5 \end{cases}$$

As it was noted in Example 10.1, all functions of the type $y = Ce^{2x}$ are solutions to the differential equation. It is not hard to see that the function

$$y = 5e^{2x}$$

is the particular solution to the given initial value problem, as it is of the form $y = Ce^{2x}$ and $y(0) = 5e^{2\cdot 0} = 5$.

10.2 Slope fields

More often than not, it is not possible to find the general solution to a differential equation. However, we would still like to determine the properties of the solution of a differential equation, e.g., zeroes, extremes or inflection points, asymptotes at infinity, etc. The existence or non-existence of such properties is generally referred to as the **behavior** of the solution.

The behavior of the solution of a initial value problem may differ, depending on the initial condition. In these cases it is of special interest to have a way to investigate this dependence.

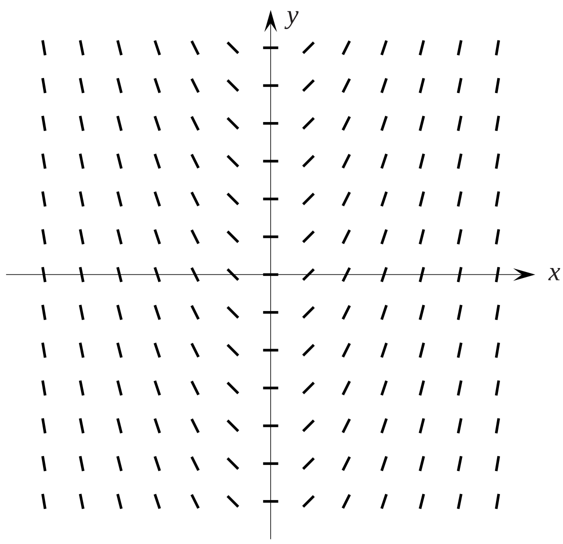
In order to visualize the behavior of the solution for different initial conditions, even when the general solution to the differential equation cannot be found, it is very convenient to use **slope fields**.

The differential equation

$$\frac{dy}{dx} = f(x,y)$$

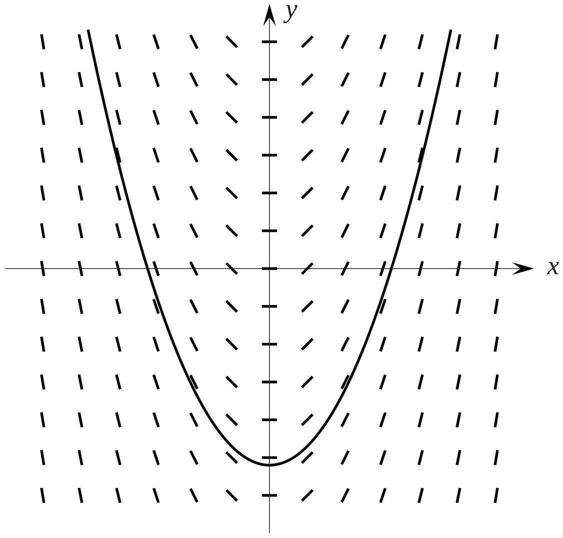
gives the derivative of the solution for any given values of x and y. According to its geometric interpretation, the derivative equals the slope of the tangent line; therefore, this value gives an idea about what the graph of the solution looks like in the neighborhood of any given point. By calculating the value of dy/dx at many points and drawing segments of the tangent line at each of them, it is possible then to get an idea of what the graph of the solution looks like in general. A diagram depicting these tangent line segments is called a **slope field**.

Example 10.4. Consider the differential equation $\dfrac{dy}{dx} = 2x$. It should be obvious that the general solution to this differential equation is $y = x^2 + C$. The slope field of this differential equation is given below:



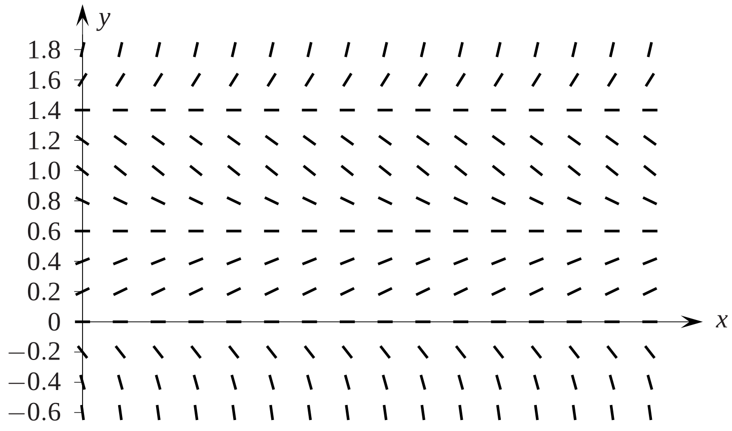
This slope field gives an idea of the behavior of the solution to the differential equation for any initial condition. For instance, it can be seen that any solution to this differential equation will have a minimum at $x = 0$, and that $\lim_{x\to\pm\infty} y(x) = \infty$.

For better understanding, we will draw the graph of a solution to this differential equation and superimpose it on the slope field:



Often it is important to determine how the initial condition affects the behavior of the solution.

Example 10.5. The slope field of a certain differential equation is shown below. Determine what the value of $\lim_{x\to\infty} y(x)$ will be, depending on the value of y_0 in the initial condition $y(0) = y_0$.

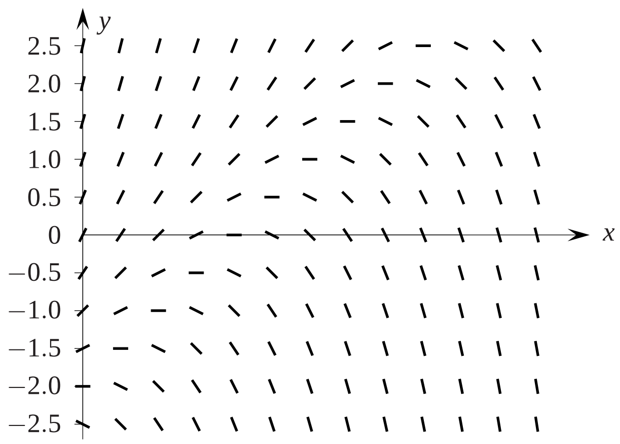


Solution. First, note that $y(x)$ does not change in either of three cases: $y = 0$, $y = 0.6$ and $y = 1.4$. If $y < 0$, however, the solution will be a decreasing function, and in this case $\lim_{x\to\infty} y(x) = -\infty$. If $0 < y < 1.4$, then $\lim_{x\to\infty} y(x) = 0.6$. Finally, if $y > 1.4$ we will have $\lim_{x\to\infty} y(x) = \infty$. Therefore, the final answer is

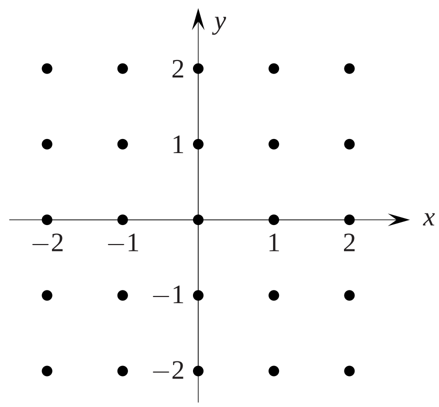
$$\lim_{x\to\infty} y(x) = \begin{cases} -\infty, & y_0 < 0; \\ 0, & y_0 = 0; \\ 0.6, & 0 < y_0 < 1.4; \\ 1.4, & y_0 = 1.4; \\ \infty, & y_0 > 1.4. \end{cases}$$

In this last example, the values $y = 0$, $y = 0.6$ and $y = 1.4$ are called **steady states**, **steady solutions**, **rest points** or simply **equilibriums**, because y does not change as $x \to \infty$ for these values. However, it can be seen that the behavior of $y(x)$ in the vicinity of these steady states is different. In particular, if $y(x)$ is *close* to 0, but not equal to it, then $\lim_{x\to\infty} y(x)$ will not equal zero—it will equal $-\infty$ or 0.6, depending on its sign. The situation at $y = 1.4$ is analogous: if y is close to 1.4, but not equal to it, then again $\lim_{x\to\infty} y(x)$ will equal either 0.6 or ∞. However, the behavior of $y(x)$ in the vicinity of 0.6 is different: if $y(x)$ is close to 0.6, then $\lim_{x\to\infty} y(x) = 0.6$. For this reason, we say that the values $y = 0$ and $y = 1.4$ are **unstable steady states**, while $y = 0.6$ is a **stable steady state**.

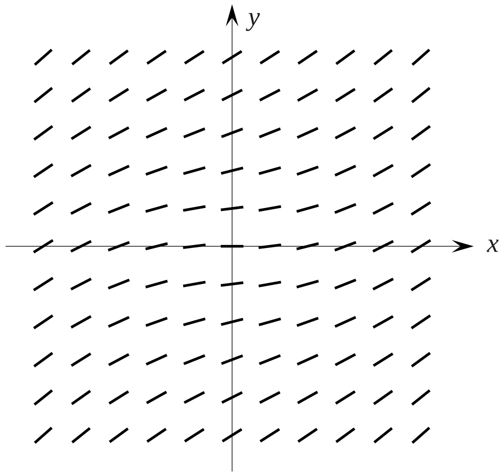
10.6. Given the initial condition $y(0) = y_0$, for what values of y_0 will the solution to the differential equation with the slope field show below have a global maximum?



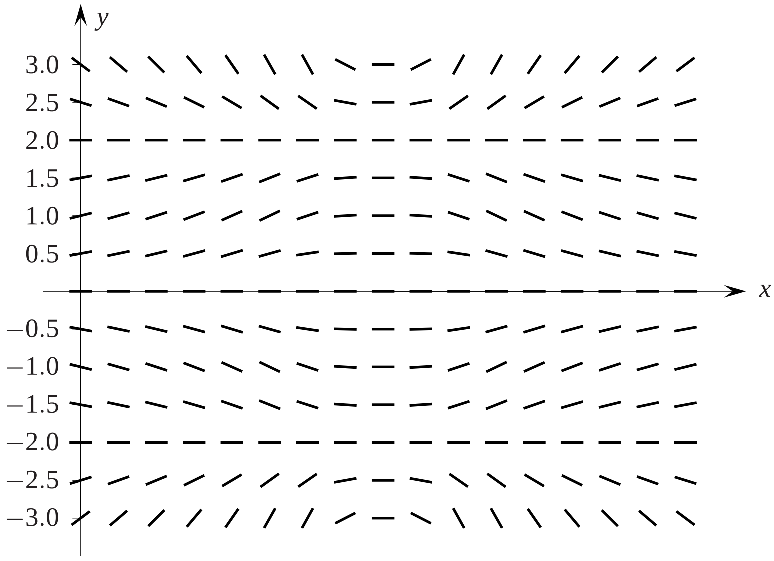
10.7. Sketch a slope field for the differential equation $\dfrac{dy}{dx} = x + y$ at the 25 points indicated below.



10.8. Consider the slope field given below. Do the solutions to the differential equation that produced this slope field have inflection points? What is the limit of these solutions as $x \to \infty$ and $x \to -\infty$?



10.9. The slope field of some differential equation with the initial condition $y(0) = y_0$ is given below. a) For what values of y_0 will the solution have a minimum? b) For what values of y_0 will the solution have a maximum? c) For what values of y_0 will the solution be stable? d) What is $\lim_{x\to\infty} y(x)$?



10.3 Separable differential equations

Definition. A **separable differential equation** is an equation of the form

$$y' = f(x)g(y).$$

Separable differential equations are special in that the variables can be *separated*—that means that they can be rewritten so that all terms involving y are on one side of the equation, and all terms involving x are on the other.

The first derivative dy/dx is considered to consist of two expressions—dx and dy, and these expressions are also separated. This means that the differentials dx and dy should be located in different sides of the equation.

Example 10.6. The differential equation $\dfrac{dy}{dx} = x$ is separable and therefore it can be solved by separating the variables:

$$dy = xdx \quad \Rightarrow \quad \int dy = \int xdx \quad \Rightarrow \quad y = \frac{x^2}{2} + C,$$

where C is an arbitrary constant.

It is important to make sure that all operations conducted during separation of variables do not change the possible values of x or y.

Example 10.7. Find the general solutions of the following differential equations:

a) $\dfrac{dy}{dx} = -\dfrac{y}{x}$; b) $(1+x)ydx + (1-y)xdy = 0$; c) $\sqrt{y^2+1}dx = xydy$.

Solution.

a) Separating the variables,

$$\frac{dy}{y} = -\frac{dx}{x} \quad \Rightarrow \quad \int \frac{dy}{y} = -\int \frac{dx}{x} \quad \Rightarrow \quad \ln|y| = -\ln|x| + C_1.$$

Note that here we assume $y \neq 0$. It is necessary to check if $y = 0$ is a solution, since otherwise it may be lost. Assuming $y = 0$, we have $y' = 0$ and the differential equation is satisfied. Therefore, $y = 0$ is a solution to the differential equation.

It is not necessary to check whether or not $x = 0$ is a solution, because the right side of the initial differential equation is not defined for $x = 0$.

Rewriting the arbitrary constant C_1 as $\ln|C_2|$, so that $C_2 \neq 0$:

$$\ln|y| = \ln\left|\frac{C_2}{x}\right| \quad \Rightarrow \quad y = \pm\frac{C_2}{x}.$$

Finally, putting $C = \pm C_2$ and remembering that $y = 0$ is a solution, we find $y = \dfrac{C}{x}$, where C is any constant, including zero.

b) Separating the variables,

$$\frac{y-1}{y}dy = \frac{1+x}{x}dx \quad \Rightarrow \quad \int\left(1 - \frac{1}{y}\right)dy = \int\left(\frac{1}{x} + 1\right)dx \quad \Rightarrow$$

$$\Rightarrow \quad y - \ln|y| = \ln|x| + x + C \quad \Rightarrow \quad y - x - \ln|xy| = C.$$

Therefore, the solution to this differential equation is an implicit function.

It is necessary to check whether any solutions were lost when we divided the differential equation by y and x. Inspection shows that $y = 0$ is a solution; while $x = 0$ satisfies the differential equation, we are interested in solutions that express y in terms of x and therefore $x = 0$ will not be considered to be a solution. The solution $y = 0$ cannot be found from the general solution $y - x - \ln|xy| = C$ for any value of C, and therefore the general solution is *only* $y - x - \ln|xy| = C$.

c) Separating the variables,

$$\frac{dx}{x} = \frac{y}{\sqrt{y^2+1}}dy \quad \Rightarrow \quad \int \frac{dx}{x} = \int \frac{y}{\sqrt{y^2+1}}dy \quad \Rightarrow \quad \ln|x| = \sqrt{y^2+1} + C_1.$$

It was necessary to divide by x. Returning to the initial differential equation, it is easy to see that $x = 0$ satisfies the equation, but as in b) it will not be considered a solution.

Continuing,

$$|x| = e^{C_1+\sqrt{y^2+1}} = C_2 e^{\sqrt{y^2+1}}, \quad C_2 > 0;$$
$$x = \pm C_2 e^{\sqrt{y^2+1}} = C_3 e^{\sqrt{y^2+1}}, \quad C_3 \neq 0.$$

Therefore, the final answer is $x = Ce^{\sqrt{y^2+1}}$, where C is an arbitrary constant.

It is of course possible to find y explicitly as a function of x in this case, but it is not absolutely necessary.

Find the solutions of the following differential equations.

10.10. $\dfrac{dy}{dx} = x$

10.11. $x^2 dx = 3y^2 dy$

10.12. $x^2 dy = y^3 dx$

10.13. $2\sqrt{x}dy = \dfrac{dx}{\sqrt{y}}$

10.14. $\dfrac{dy}{\sqrt{x}} = \dfrac{3dx}{y}$

10.15. $(1+2y)dx = (1+x^2)dy$

10.16. $x\cos^2 y dx = (1+x^2)dy$

10.17. $y^2 dx + (2x+4)dy = 0$

10.18. $(x^3+yx^3)dy = (xy^2 - x^2y^2)dx$

10.19. $x^2 dy - (2xy+3y)dx = 0$

10.20. $(1+x^2)dy - \sqrt{y}dx = 0$

10.21. $\sqrt{1-x^2}y' = x\sqrt{1-y^2}$

Example 10.8. Find the particular solutions of the following initial value problems:

$$\text{a) } \frac{dy}{dx} = 4y,\ y(2) = e^{10}; \quad \text{b) } y' = \frac{1+y^2}{1+x^2},\ y(0) = 1$$

Solution.

a) It is first necessary to find the general solution. Separating the variables,

$$\frac{dy}{y} = 4dx \quad \Rightarrow \quad \int \frac{dy}{y} = \int 4dx \quad \Rightarrow \quad \ln|y| = 4x + C_1.$$

Note that $y = 0$ is a solution to the initial differential equation.

$$|y| = e^{C_1+4x} = C_2 e^{4x}, \quad C_2 > 0;$$
$$y = \pm C_2 e^{4x} = C_3 e^{4x}, \quad C_3 \neq 0.$$

Recalling that $y = 0$ satisfies the differential equation, the general solution can be written as $y = Ce^{4x}$, where C is arbitrary.

The initial condition gives $e^{10} = Ce^8$, so $C = e^2$; the final answer is $y = e^{2+4x}$.

b) Separating the variables,

$$\frac{dy}{1+y^2} = \frac{dx}{1+x^2} \quad \Rightarrow \quad \int \frac{dy}{1+y^2} = \int \frac{dx}{1+x^2} \quad \Rightarrow \quad \tan^{-1} y = \tan^{-1} x + C.$$

Note that it is easiest to find C from the initial condition at this point: $\dfrac{\pi}{4} = 0 + C$, so $C = \dfrac{\pi}{4}$; the final answer is $y = \tan\left(\tan^{-1} x + \dfrac{\pi}{4}\right)$.

Find the particular solutions of the following initial value problems.

10.22. $\frac{dy}{dx} - \frac{y}{x} = 0,\ y(2) = 4$

10.23. $ydy - xdx = 0,\ y(-1) = 3$

10.24. $\frac{dy}{dx} = -2y,\ y|_{x=2} = e$

10.25. $yy' = \frac{1-2x}{y},\ y(-2) = 1$

10.26. $xy' = 2 + x^2,\ y|_{x=1} = 3$

10.27. $2\sqrt{x}dy = ydx,\ y(4) = 1$

10.28. $\frac{dy}{dx} = 2\sqrt{y}\ln x,\ y(e) = 1$

10.29. $(1 + e^{2x})y' = (1 + y^2)e^x,\ y|_{x=0} = 0$

10.30. $y'\sin x = y\ln y,\ y\left(\frac{\pi}{2}\right) = e$

10.31. $\sin^2 x dy = \cos^2 y dx,\ y(3) = \frac{\pi}{2}$

10.32. $\sin y \cos x dy = \cos y \sin x dx,\ y(0) = \frac{\pi}{4}$

10.33. The diameter of a circle is decreasing at a rate equal to the area of the circle at any given moment. Find the diameter D of the circle as a function of time t if D was equal to 1 cm at time $t = 0$.

10.34. An object starts from the point $M(4,0)$ and moves along the x-axis so that its velocity is equal to $v = 2t + 3t^2$ for any $t \geq 0$. Find the position of the object for any value of t.

10.35. The population of a bacteria colony doubles every 2 days. Find the population of the colony after 5 days of growth if the initial population was 2000.

10.36. A bacterial culture grows at a rate proportional to the number of bacteria present. If the size of the culture triples every 9 hours, how long does it take for the culture to double?

10.37. The weight of a puppy in the first 3 months of life doubles every 20 days. If the weight of a newborn puppy is 200 grams, how old is a puppy that weighs 300 grams? How much will a puppy weigh 70 days after birth?

10.38. Elasticity is defined as $\varepsilon = \frac{dQ}{dP}\frac{P}{Q}$, where P is price and Q is demand. Find the demand function $Q = f(P)$ if a) $\varepsilon = -1$ and $Q(5) = 200$; b) $\varepsilon = \frac{-5P+2P^2}{Q}$ and $Q(10) = 500$.

10.39. Radium decays at a rate proportionate to its amount. The half-life of radium (i.e., the time needed for half of the radium to decay) is 1600 years. Determine what percent of the radium will decay after 100 years.

10.40. The remains of an ancient campfire are unearthed and it is found that there is only 80% as much radioactive carbon-14 in the charcoal samples from the campfire as there is in modern living trees. If the half-life of carbon-14 is 5730 years, how long ago did the campfire burn?

10.41. The population $P(t)$ of wolves in a national park is increasing at a rate proportionate to $800 - P(t)$, where t is measured in years and the constant of proportionality is k. In how many years will the population of wolves reach 700 if $P(0) = 300$ and $P(5) = 400$?

10.42. During a zero-order chemical reaction, the concentration of a substance decreases at a constant rate. If two thirds of the substance reacts within the first hour, how long will it take for the concentration of the substance to decrease to 10% of its initial value?

10.43. During a first-order chemical reaction, the concentration of a substance decreases at a rate proportionate to itself. If two thirds of the substance reacts within the first hour, how long will it take for the concentration of the substance to decrease to 10% of its initial value?

10.44. During some catalytic chemical reactions, the concentration of a substance decreases at a rate proportionate to itself raised to the power of 1.5. If two thirds of the substance reacts within the first hour, how long will it take for the concentration of the substance to decrease to 10% of its initial value?

10.45. During a second-order chemical reaction, the concentration of a substance decreases at a rate proportionate to its square. If two thirds of the substance reacts within the first hour, how long will it take for the concentration of the substance to decrease to 10% of its initial value?

10.46. During a third-order chemical reaction, the concentration of a substance decreases at a rate proportionate to its cube. If two thirds of the substance reacts within the first hour, how long will it take for the concentration of the substance to decrease to 10% of its initial value?

10.47. When sugar is dissolved in water, it dissolves at a rate proportionate to the amount of undissolved sugar present. After 1 minute, 75% of an initial portion of sugar is still in the form of crystals. How long does it take for 75% of an initial portion of sugar to dissolve? After 150 seconds there are 10 grams of undissolved sugar left. How much sugar was there initially?

10.48. A 25 year old man is given \$50,000 which is invested at 5% per year, compounded continuously. He intends to deposit money continuously at the rate of \$2,000 per year. Assuming that the interest rate remains at 5%, the amount of money $A(t)$ at time t satisfies the equation $A' = 0.05A + 2000$. a) Determine the amount of money in the account when the man is 65. b) At age 65, he will stop depositing money and start withdrawing money continuously at the rate of W dollars per year. If the money must last until the man is 85, what is the largest possible value of W?

10.4 Homogeneous differential equations

Definition. A **homogeneous** differential equation is one that can be written in the form

$$y' = F\left(\frac{y}{x}\right).$$

In other words, the right side depends not so much on y and x, as it depends on the *ratio* of y and x.

An equivalent definition of homogeneous differential equations are that, given $y' = F(x,y)$, we have $F(tx,ty) = F(x,y)$ for any t.

Homogeneous differential equations can be solved by substitution, introducing a new unknown function $u(x)$ and putting $y(x) = xu(x)$.

Example 10.9. Solve the equation $x^2y' = xy + y^2$.

Solution. First, we rewrite the equation as

$$y' = \frac{y}{x} + \left(\frac{y}{x}\right)^2.$$

Since the right side of the equation can be considered to depend only on the ratio $\frac{y}{x}$ (rather than on x and y separately), this equation is homogeneous.

The substitution $y = xu(x)$ implies $y' = u + xu'$; thus the differential equation becomes

$$u + xu' = u + u^2,$$

or simply $xu' = u^2$. The equation is now separable; using the method given in section 10.3, we find $u = \frac{1}{C - \ln|x|}$, so the final answer is

$$y = \frac{x}{C - \ln|x|}.$$

Example 10.10. Solve the differential equation $y' = \frac{5y-2x}{y+2x}$.

Solution. Since this is a homogeneous differential equation, put $y = xu(x)$; we will find

$$xu' + u = \frac{5u-2}{u+2},$$

or simply

$$u' = -\frac{1}{x} \cdot \frac{u^2 - 3u + 2}{u+2}.$$

This is a separable differential equation. After separating the variables and integrating (we will need to integrate a rational function, see section 9.4), we will find three possible solutions for u:

a) $-4\ln|u-2| + 3\ln|u-1| = \ln|x| + C$;

b) $u = 1$;

c) $u = 2$.

In terms of y, the solutions are

a) $-4\ln|y-2x| + 3\ln|y-x| = C$;

b) $y = x$;

c) $y = 2x$.

Find the solutions of the following homogeneous differential equations:

10.49. $xy' = x + y$

10.50. $y' = 5 + 2\frac{y}{x}$

10.51. $\left(x^2 - y^2\right)dx + xydy = 0$

10.52. $y' = \frac{x^2+y^2}{2x^2}$

10.53. $2(x+2y)dx + (y-x)dy = 0$

10.54. $y' = \frac{xy}{x^2+y^2}$

10.55. $(x+y)dy + (x-y)dx = 0$

10.56. $\left(y - \sqrt{x^2 - 4y^2}\right)dx - xdy = 0$

10.5 Linear differential equations

Definition. A **linear differential equation** is an equation of the form

$$y' + f(x)y = g(x).$$

If $g(x)$ is identically equal to zero, then the equation is separable.

Linear differential equations can be solved by any of several equivalent methods. We will discuss two of them.

Example 10.11. Solve the equation $y' + xy = x$.

Solution.

Method 1.

Suppose $y(x) = u(x)v(x)$ (any function can be written as the product of two other functions). We have $y' = u'v + v'u$, therefore $u'v + v'u + xuv = x$, or

$$u'v + (v' + xv)u = x.$$

We can choose the function v almost any way we wish; the function u will then have to be chosen as to make y satisfy the differential equation. If we choose $v' + xv = 0$, then we have $v = Ce^{-x^2/2}$; we choose $C = 1$, so that now

$$v = e^{-x^2/2}.$$

The differential equation can now be written as

$$e^{-x^2/2}u' = x,$$

which can be solved by separation of variables; the answer is $u = e^{x^2/2} + C$. Thus, the final answer is

$$y = uv = \left(e^{x^2/2} + C\right) e^{-x^2/2} = Ce^{-x^2/2} + 1.$$

Method 2

Consider first the homogeneous equation $y' + xy = 0$. This equation is separable; the solution is $y = Ce^{-x^2/2}$. In order to find the solution to the differential equation, we will attempt to find it in the form $y = C(x)e^{-x^2/2}$. We have $y' = C'(x)e^{-x^2/2} - xC(x)e^{-x^2/2}$, so

$$C'(x)e^{-x^2/2} - xC(x)e^{-x^2/2} + xC(x)e^{-x^2/2} = x.$$

Therefore, $C'(x) = xe^{x^2/2}$ and $C(x) = e^{x^2/2} + C$. The final answer is

$$y = \left(e^{x^2/2} + C\right) e^{-x^2/2} = 1 + Ce^{-x^2/2}.$$

Find the solutions of the following linear differential equations:

10.57. $y' + xy = x$

10.58. $y' + \frac{y}{x} = x$

10.59. $y' + \frac{2}{x}y = 5x^2 - 3$

10.60. $y' + 2y = e^x$

10.61. $xy' + 2y = x^2 - x + 1$

10.62. $2xy' - y = x + 1$

10.63. $2y' - y = 4\sin 3x$

10.64. $y' + (\tan x)y = \cos^2 x$

10.65. $y' = \frac{2x}{1+x^2}y + \frac{2}{1+x^2}$

10.66. $(\cos^2 x)(\sin x)y' = -(\cos^3 x)y + 1$

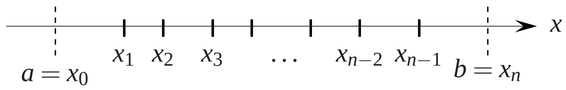
10.67. $(\cos x)y' + (\sin x)y = 2\cos^3 x \sin x - 1$

Chapter 11.
THE DEFINITE INTEGRAL

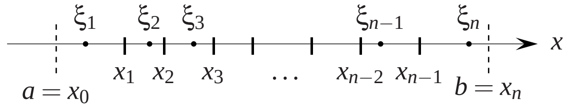
11.1 Riemann sums and the definite integral

Consider a *continuous* function $f(x)$ on the interval $[a,b]$.

1. Divide the interval $[a,b]$ into n subintervals. (Note that these subintervals are not necessarily equal.) Denote the left endpoint of the i-th subinterval by x_{i-1}, and the right endpoint by x_i. The length of the i-th subinterval is therefore equal to $\triangle x_i = x_i - x_{i-1}$.



2. Choose in an arbitrary manner a number ξ_i in each interval.



3. Calculate the value of $f(\xi_i)\triangle x_i$ for all subintervals, the product of $f(\xi_i)$ and the length of the corresponding subinterval.

4. Find the sum of all $f(\xi_i)\triangle x_i$, which we will write as $\sum\limits_{i=1}^{n} f(\xi_i)\triangle x_i$. This expression is called a **Riemann sum**, or an **integral sum**.

Consider now a process in which the number of divisions increases without bound ($n \to \infty$), and in which the maximum subinterval length decreases to zero ($\max \triangle x_i \to 0$). If the limit of the Riemann sum in this process does not depend on how the numbers ξ_i where chosen on each subinterval, or how $[a,b]$ was divided into subintervals, then this limit is called the **definite integral** of $f(x)$ on the interval $[a,b]$ and denoted

$$\int\limits_a^b f(x)dx = \lim_{\substack{n \to \infty \\ \max \triangle x_i \to 0}} \sum_{i=1}^{n} f(\xi_i)\triangle x_i.$$

Theorem. If $f(x)$ is continuous on $[a,b]$, then $\int\limits_a^b f(x)dx$ exists.

The **geometric interpretation** of the definite integral is that if $f(x) \geq 0$ for all $x \in [a,b]$, then $\int_a^b f(x)dx$ equals the area of the region bounded by the graph of $y = f(x)$, the x-axis, and the vertical lines $x = a$ and $x = b$.

If $f(x) \leq 0$ for all $x \in [a,b]$, then $-\int_a^b f(x)dx$ equals the area of the region bounded by the graph of $y = f(x)$, the x-axis, and the vertical lines $x = a$ and $x = b$.

Example 11.1. Find the value of $\int_0^1 (x+1)dx$ using the definition.

First of all, divide the interval $[0,1]$ into n *equal* subintervals. This means that the length of each subinterval will equal $\triangle x = 1/n$.

Next, choose the points ξ_i so that they coincide with the right endpoint of each subinterval: $\xi_i = i/n$. We will have

$$f(\xi_i) = \xi_i + 1 = \frac{i}{n} + 1.$$

Now construct the Riemann sum and find its limit as $n \to \infty$. Since all subintervals were chosen to be equal in length, the condition $\max \triangle x_i \to 0$ will be satisfied automatically.

$$\int_0^1 (x+1)dx = \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{i}{n} + 1\right)\frac{1}{n}.$$

The sum can be simplified:

$$\sum_{i=1}^{n} \left(\frac{i}{n} + 1\right)\frac{1}{n} = \frac{1}{n^2}\sum_{i=1}^{n} i + \frac{1}{n}\sum_{i=1}^{n} 1 = \frac{n(n+1)}{2n^2} + 1.$$

Here we used the formula for the sum of an arithmetic progression. Now find the limit:

$$\int_0^1 (x+1)dx = \lim_{n\to\infty} \left(\frac{n(n+1)}{2n^2} + 1\right) = \frac{1}{2} + 1 = \frac{3}{2}.$$

11.1. The expression

$$\frac{1}{25}\left(\sqrt{\frac{1}{25}} + \sqrt{\frac{2}{25}} + \ldots + \sqrt{\frac{25}{25}}\right)$$

is an approximation for what integral?

Find the following integrals using the definition.

11.2. $\int_1^3 x dx$

11.3. $\int_{-2}^3 (x-2)dx$

11.4. $\int_a^b \frac{dx}{x^2}$.

11.5. $\int_0^1 e^x dx$

11.6. $\int_1^2 \frac{dx}{x}$

11.7. $\int_0^{\pi/2} \sin x dx$.

Approximate calculation of definite integrals

The *approximate* value of an integral can be found by calculating the value of the Riemann sum for a fixed number of subintervals. This requires choosing:

1. How the interval $[a,b]$ was divided into subintervals;
2. How the numbers ξ_i were chosen on each subinterval.

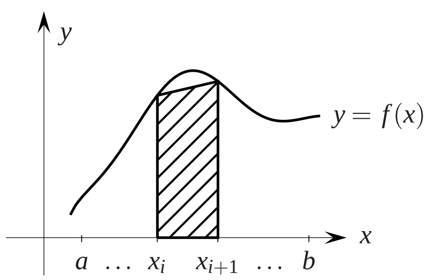
Usually the method for dividing $[a,b]$ into subintervals is given. There are several standard ways of choosing ξ_i:

- Leftpoint sums: ξ_i is chosen to be the left endpoint of each interval.
- Rightpoint sums: ξ_i is chosen to be the right endpoint of each interval.
- Midpoint sums: ξ_i is chosen to be the midpoint of each interval.
- Inscribed rectangles: ξ_i is chosen so that $f(\xi_i)$ is the minimal value of $f(x)$ on the i-th subinterval.
- Circumscribed rectangles: ξ_i is chosen so that $f(\xi_i)$ is the maximal value of $f(x)$ on the i-th subinterval.

A better approximation to $\int_a^b f(x)dx$ can be found using one of the following methods:

Trapezoidal approximation

Trapezoids can be used instead of using rectangles to increase the accuracy of approximating the value of a definite integral. Instead of building a rectangle on each subdivision, it is possible to build trapezoids. The area (assuming that $f(x) > 0$) of each trapezoid is equal to $\frac{f(x_{i+1})+f(x_i)}{2}(x_{i+1}-x_i)$.



Therefore, we have

$$\int_a^b f(x)dx \approx \frac{f(a)+f(x_1)}{2}(x_1-a)+\frac{f(x_1)+f(x_2)}{2}(x_2-x_1)+\ldots+$$
$$+\frac{f(x_{n-1})+f(b)}{2}(b-x_{n-1}).$$

If the length of all the subdivisions are equal, i.e. $x_1-a=x_2-x_1=\ldots=b-x_{n-1}=\triangle x$, then this expression can be simplified:

$$\int_a^b f(x)dx \approx \left(\frac{f(a)}{2}+f(x_1)+f(x_2)+\ldots+f(x_{n-1})+\frac{f(b)}{2}\right)\triangle x.$$

Simpson's rule

It is possible to increase the accuracy of approximation by using parabolas rather than rectangles or trapezoids. In this case it is absolutely necessary that the subintervals be equal, and that there is an *even* number of them. For instance, if there are two subintervals, then

$$\int_a^b f(x)dx \approx \frac{h}{3}\left(f(a)+4f(x_1)+f(b)\right),$$

where $x_1=\frac{a+b}{2}$ and h is the length of each subinterval ($h=\frac{b-a}{2}$). This approximation is known as **Simpson's rule with three ordinates (or with two divisions)**.

If there are four subintervals, then

$$\int_a^b f(x)dx \approx \frac{h}{3}\left(f(a)+4f(x_1)+2f(x_2)+4f(x_3)+f(b)\right),$$

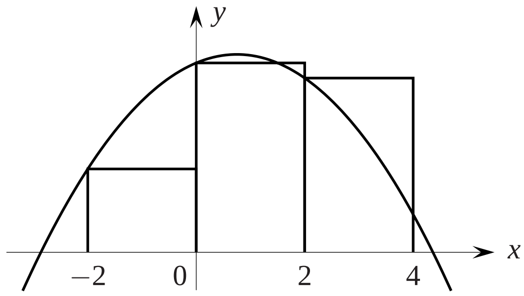
where x_1, x_2 and x_3 divide the interval $[a,b]$ into four equal subintervals and h is the length of each subinterval; this formula is known as **Simpson's rule with five ordinates (or with four divisions)**.

Simpson's rule can be extended to any even number of equal subintervals.

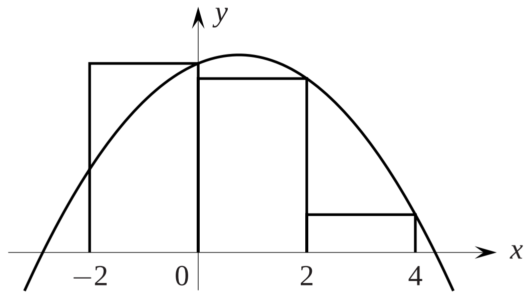
Example 11.2. Find the approximate value of $\int_{-2}^{4}(3x-2x^2+25)dx$ using 3 equal subintervals and a) left sums; b) right sums; c) midpoint sums; d) inscribed rectangles; e) circumscribed rectangles.

Solution. The 3 subintervals are $[-2,0]$, $[0,2]$ and $[2,4]$; the length of each subinterval is 2.

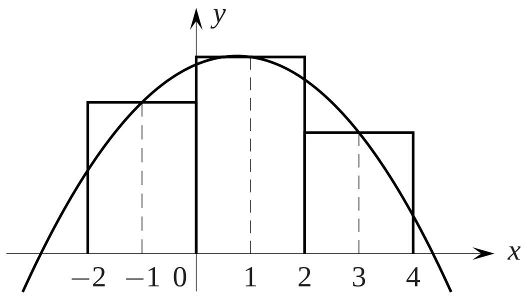
a) $\displaystyle\int_{-2}^{4}(3x-2x^2+25)dx \approx$
$\approx \big(f(-2)+f(0)+f(2)\big)\,2 =$
$= (11+25+23)2 = 118\,.$



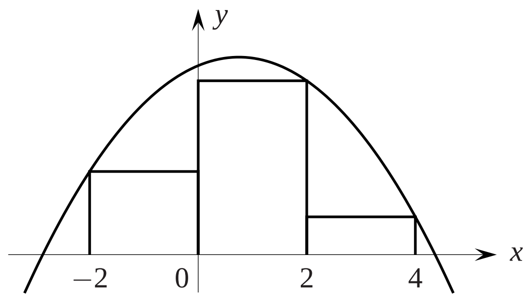
b) $\displaystyle\int_{-2}^{4}(3x-2x^2+25)dx \approx$
$\approx \big(f(0)+f(2)+f(4)\big)\,2 =$
$= (25+23+5)2 = 106\,.$



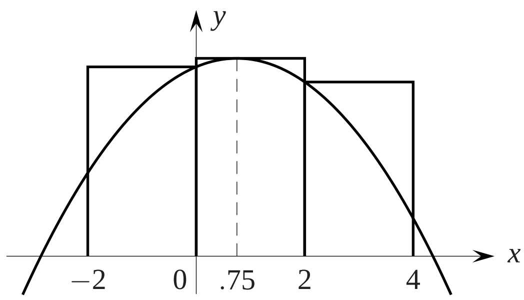
c) $\displaystyle\int_{-2}^{4}(3x-2x^2+25)dx \approx$
$\approx \big(f(-1)+f(1)+f(3)\big)\,2 =$
$= (20+26+16)2 = 124\,.$



d) $\displaystyle\int_{-2}^{4}(3x-2x^2+25)dx \approx$
$\approx \big(f(-2)+f(2)+f(4)\big)\,2 =$
$= (11+23+5)2 = 78\,.$



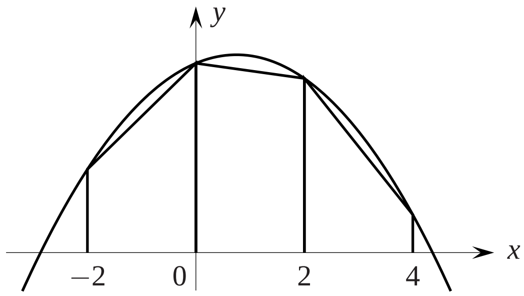
e) Note that the maximum value of $f(x)$ on the interval $[0,2]$ is attained at $x = 3/4$.
$\displaystyle\int_{-2}^{4}(3x-2x^2+25)dx \approx$
$\approx \big(f(0)+f(.75)+f(2)\big)\,2 =$
$= (25+26.125+23)2 = 148.25\,.$



Example 11.3. Find the approximate value of $\displaystyle\int_{-2}^{4}(3x-2x^2+25)dx$ using the trapezoidal rule for 3 equal subintervals.

Solution. The 3 subintervals are $[-2,0]$, $[0,2]$ and $[2,4]$; the length of each subinterval is 2.

$$\int_{-2}^{4}(3x-2x^2+25)dx \approx$$
$$\approx \left(\frac{f(-2)}{2}+f(0)+f(2)+\frac{f(4)}{2}\right)2 =$$
$$= \big(5.5+25+23+2.5\big)2 = 112.$$



Note that the result for the trapezoidal rule is the arithmetic mean of the left sum approximation and the right sum approximation found in the previous example.

Example 11.4. Find the approximate value of $\int_{-2}^{4}(4x^4-30x^2)dx$ using Simpson's rule with a) 4 divisions and b) 6 divisions.

Solution. a) Dividing the interval $[-2,4]$ into 4 equal subdivisions can be achieved by putting $a=-2$, $x_1=-0.5$, $x_2=1$, $x_3=2.5$ and $b=4$; the length of each subinterval is 1.5. The approximate value of the integral is

$$\int_{-2}^{4}(4x^4-30x^2)dx \approx \frac{1.5}{3}\big(f(-2)+4f(-0.5)+2f(1)+4f(2.5)+f(4)\big) =$$
$$=\frac{1}{2}(-56-29-52-125+544) = 141.$$

b) Dividing the interval $[-2,4]$ into 6 equal subdivisions will require a subdivision length of 1, and we will have $a=-2$, $x_1=-1$, $x_2=0$, $x_3=1$, $x_4=2$, $x_5=3$ and $b=4$. The approximate value of the integral using this method will be

$$\int_{-2}^{4}(4x^4-30x^2)dx \approx \frac{1}{3}\big(f(-2)+4f(-1)+2f(0)+4f(1)+2f(2)+4f(3)+f(4)\big) =$$
$$=\frac{1}{3}\big(-56-104+0-104-112+216+544\big) = 128.$$

Note that the accuracy increases with the number of subintervals; the actual value of the integral is 124.8.

11.8. Find the approximate area between the curve $f(x)=x^3-x+1$ and the x-axis on the interval $[0,2]$ using 4 intervals of equal length and a) left sums; b) right sums; c) midpoint sums.

11.9. Find the approximate area between the curve $f(x)=x^3-x+1$ and the x-axis on the interval $[0,2]$ using 4 intervals of equal length and the trapezoidal rule.

11.10. Knowing that $\int_{0}^{1/2}\frac{dx}{\sqrt{1-x^2}}=\frac{\pi}{6}$, find the approximate value of π using 4 equal subintervals and the trapezoidal rule.

11.11. Find the approximate area between the curve $f(x)=\sqrt{3+\cos x}$ and the x-axis on the interval $[0,\pi]$ using 6 intervals of equal length and a) left sums; b) right sums; c) midpoint sums; d) the trapezoidal rule.

11.12. A table of values for a continuous function $f(x)$ is given below. Find the trapezoidal approximation of $\int_0^2 f(x)dx$ using four equal subintervals.

x	0	0.5	1.0	1.5	2.0
$f(x)$	3	3	5	8	13

11.13. A 100-foot pond's width was measured at 10-foot intervals. The results are given in the table below. Find the approximate surface area of the pond using the trapezoidal rule.

No.	1	2	3	4	5	6	7	8	9	10	11
Width, ft	0	195	191	174	106	97	121	138	147	141	0

11.2 Calculation of definite integrals

Properties of the definite integral

1. $\int_a^b kf(x)dx = k\int_a^b f(x)dx$, where k is a constant;

2. $\int_a^b \left(f(x)+g(x)\right)dx = \int_a^b f(x)dx + \int_a^b g(x)dx$;

3. $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$;

4. $\int_a^b f(x)dx = -\int_b^a f(x)dx$;

5. If $f(x) \le g(x)$ for all $x \in [a,b]$, then $\int_a^b f(x)dx \le \int_a^b g(x)dx$.

Theorem (Mean value theorem for integrals)

If $f(x)$ is continuous, then there exists $c \in [a,b]$ such that

$$f(c) = \frac{1}{b-a}\int_a^b f(x)dx.$$

$f(c)$ is called the **average** or **mean value** of $f(x)$ over the interval $[a,b]$.

Theorem (Variable upper bound)

The derivative of the function given by $F(x) = \int\limits_a^x f(t)dt$ is

$$F'(x) = f(x).$$

Theorem (The fundamental theorem of calculus)

If $f(x)$ is continuous on the interval $[a,b]$ and $F(x)$ is an antiderivative of $f(x)$, then

$$\int\limits_a^b f(x)dx = F(b) - F(a).$$

Example 11.5. Find a) $\int\limits_{1/e}^{1/\sqrt{e}} \frac{dx}{x}$; b) $\int\limits_0^4 \left(3x - e^{x/4}\right) dx$.

Solution.

a) $$\int\limits_{1/e}^{1/\sqrt{e}} \frac{dx}{x} = \ln|x| \Bigg|_{1/e}^{1/\sqrt{e}} = \ln\left(\frac{1}{\sqrt{e}}\right) - \ln\left(\frac{1}{e}\right) = -\frac{1}{2} + 1 = \frac{1}{2}.$$

b) $$\int\limits_0^4 \left(3x - e^{x/4}\right) dx = 3\frac{x^2}{2}\Bigg|_0^4 - 4e^{x/4}\Bigg|_0^4 = 3(8-0) - 4(e-1) = 28 - 4e.$$

Substitution can also be used to calculate definite integrals, and the technique (i.e. choosing what substitution to make) is the same as for indefinite integrals. It is of paramount importance to remember that the limits of the integral must be changed as well. The formula for substitution in a definite integral is:

$$\int\limits_a^b f(x)dx = \int\limits_\alpha^\beta f(\phi(t))\,\phi'(t)dt,$$

where $x = \phi(t)$ is a continuously differentiable function defined on $[\alpha, \beta]$ and $\phi(\alpha) = a$, $\phi(\beta) = b$.

Examples of using substitution to find definite integrals is shown in the examples below.

Example 11.6. Find a) $\int\limits_{-1}^{6} \frac{dx}{\sqrt{3x+7}}$; b) $\int\limits_{15}^{99} \frac{dx}{3 - \sqrt{x+1}}$.

Solution.

a) $$\int_{-1}^{6}\frac{dx}{\sqrt{3x+7}}=\left\{\begin{matrix} t=3x+7 & \Rightarrow & dx=\frac{1}{3}dt; \\ x=-1 & \Rightarrow & t=4; \\ x=6 & \Rightarrow & t=25 \end{matrix}\right\}=\frac{1}{3}\int_{4}^{25}\frac{dt}{\sqrt{t}}=$$

$$=\frac{2}{3}\sqrt{t}\Big|_{4}^{25}=\frac{2}{3}(5-2)=2.$$

b) $$\int_{15}^{99}\frac{dx}{3-\sqrt{x+1}}=\left\{\begin{matrix} t^2=x+1 & \Rightarrow & dx=2tdt; \\ x=15 & \Rightarrow & t=4; \\ x=99 & \Rightarrow & t=10 \end{matrix}\right\}=\int_{4}^{10}\frac{2t}{3-t}dt=$$

$$=-2\int_{4}^{10}\frac{t-3+3}{t-3}dt=-2\int_{4}^{10}\left(1+\frac{3}{t-3}\right)dt=-2t\Big|_{4}^{10}-6\ln|t-3|\Big|_{4}^{10}=$$

$$=-2(10-4)-6(\ln 7-0)=-12-6\ln 7.$$

Definite integrals can also be integrated by parts, using the formula

$$\int_{a}^{b}udv=uv\Big|_{a}^{b}-\int_{a}^{b}vdu.$$

Integration by parts is used for finding definite integral in the same way as for finding indefinite integrals.

Example 11.7. Find $\int_{0}^{5}xe^{x}dx$.

Solution.

$$\int_{0}^{5}xe^{x}dx=\left\{\begin{matrix} u=x, & du=dx \\ dv=e^{x}dx, & v=e^{x} \end{matrix}\right\}=xe^{x}\Big|_{0}^{5}-\int_{0}^{5}e^{x}dx=5e^{5}-0-e^{x}\Big|_{0}^{5}=$$

$$=5e^{5}-e^{5}+1=4e^{5}+1.$$

Find the following definite integrals by substitution and by integration by parts.

11.14. $\int_{2}^{3}3x^{2}dx.$

11.15. $\int_{0}^{1}(2+e^{x/2})dx.$

11.16. $\int_{0}^{3}\frac{dx}{x^{2}-16}.$

11.17. $\int_{0}^{1}\frac{xdx}{4+x^{2}}.$

11.18. $\int_{-3}^{-2}\frac{dx}{(4+3x)^{3}}.$

11.19. $\int_{0}^{1}\sqrt{2+x}dx.$

11.20. $\int\limits_0^4 \sqrt[3]{4x+8}dx.$

11.21. $\int\limits_{1/\pi}^{2/\pi} \frac{\cos(\frac{1}{x}+\frac{\pi}{4})}{x^2}dx.$

11.22. $\int\limits_0^{1/2} \frac{dx}{\sqrt{1-x^2}}.$

11.23. $\int\limits_0^{1/2} \frac{xdx}{\sqrt{1-x^2}}.$

11.24. $\int\limits_0^{1/2} \frac{x^2dx}{\sqrt{1-x^2}}.$

11.25. $\int\limits_0^{1/2} \frac{x^3dx}{\sqrt{1-x^2}}.$

11.26. $\int\limits_0^{1/2} \frac{x^4dx}{\sqrt{1-x^2}}.$

11.27. $\int\limits_0^2 x^5\sqrt{1+x^3}dx.$

11.28. $\int\limits_0^1 e^{2x}(e^{2x}+1)^3dx.$

11.29. $\int\limits_3^8 \frac{dx}{1+\sqrt{x+1}}.$

11.30. $\int\limits_{-2}^2 \frac{dx}{\sqrt{x^2+2x+2}}.$

11.31. $\int\limits_1^2 \frac{xdx}{\sqrt{2x+1}}.$

11.32. $\int\limits_0^{\pi/4} \cos^2\left(x+\frac{\pi}{4}\right)dx.$

11.33. $\int\limits_{\pi/6}^{\pi/3} \cos 4x\cos 5xdx.$

11.34. $\int\limits_1^2 \frac{xdx}{64-x^4}.$

11.35. $\int\limits_0^{\pi} x\sin xdx.$

11.36. $\int\limits_0^{\sqrt{3}} \tan^{-1}xdx.$

11.37. $\int\limits_{-\pi/2}^0 e^{x/\pi}\cos xdx.$

11.38. $\int\limits_2^4 x\ln 2xdx.$

11.39. $\int\limits_{1/2}^{1/\sqrt{2}} \sqrt{1-x^2}dx.$

11.40. $\int\limits_{\pi/12}^{\pi/6} \frac{xdx}{\cos^2(2x)}.$

11.41. $\int\limits_0^1 \frac{dx}{\sqrt{x}+\sqrt[3]{x}}.$

11.42. $\int\limits_{1/2}^{\sqrt{3}/2} \sin^{-1}xdx.$

11.43. $\int\limits_{-2}^0 \frac{e^x+3}{e^x+4}dx.$

11.44. $\int\limits_{-5/3}^1 \frac{\sqrt{x+2}}{x+3}dx.$

11.45. (Simpson's rule) Prove that the integral of

$$f(x) = c_2x^2+c_1x+c_0,$$

over the interval $[a,b]$ is equal to

$$A = \frac{b-a}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right].$$

11.46. Let $f(x)$ be a continuous function that is defined for all real numbers x and that has the following properties:

$$\int_1^3 f(x)dx = \frac{5}{2}; \qquad \int_1^5 f(x)dx = 10.$$

a) Find the average (mean) value of $f(x)$ over the closed interval $[1,3]$. b) Find the value of $\int_3^5 (2f(x)+6)dx$. c) Given that $f(x) = ax+b$, find a and b.

Calculate the following limits using definite integrals.

11.47. $\lim\limits_{n\to\infty} \frac{1}{\sqrt{n^3}} \sum\limits_{k=1}^{n} \sqrt{n+k}$.

11.48. $\lim\limits_{n\to\infty} \sum\limits_{k=1}^{n} \frac{k}{n^2+k^2}$.

11.49. $\frac{\pi}{3} \lim\limits_{n\to\infty} \frac{1}{n^2} \sum\limits_{k=1}^{n} k\cos\left(\frac{\pi k^2}{3n^2}\right)$.

11.50. $3\lim\limits_{n\to\infty} \sqrt{\frac{3}{n}} \sum\limits_{k=1}^{n} \frac{\sqrt{k}}{3k+n}$.

11.51. $\lim\limits_{n\to\infty} \frac{\sqrt[n]{(n+1)(n+2)\cdot\ldots\cdot(2n-1)(2n)}}{n}$.

11.52. Determine whether or not the integral $\int_{-2}^{2} x^{11}11^x dx$ is positive or negative. Justify your answer.

11.53. Prove that

$$\int_a^b |f(x)|\,dx \geq \left|\int_a^b f(x)dx\right|.$$

Find the following limits:

11.54. $\lim\limits_{x\to 0} \frac{1}{x}\int_0^x \frac{dt}{\sqrt{1+t^4}}$.

11.55. $\lim\limits_{x\to 0} \frac{1}{x}\int_{1/x}^{2/x} \frac{dt}{t^2}$.

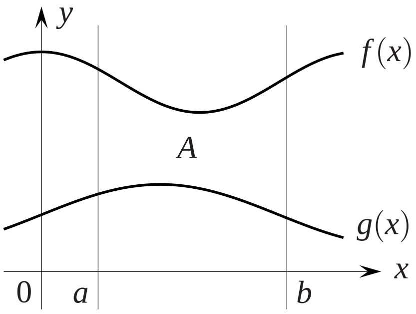
11.56. $\lim\limits_{x\to 1} \frac{\int_1^x \frac{\sin t}{t}dt}{\int_1^x \frac{e^t}{t}dt}$.

11.57. $\lim\limits_{x\to 0} \frac{\int_0^x t\sin(\sin t)dt}{\int_0^x \ln^2(1+t)dt}$.

11.3 Area of regions on the coordinate plane

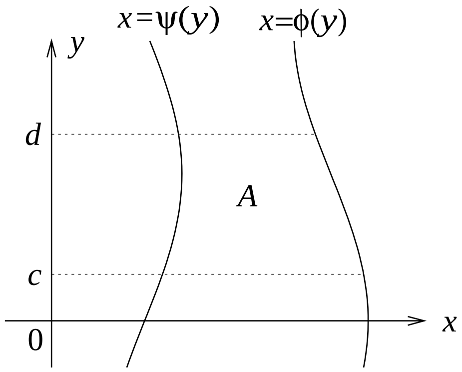
The area of regions on the coordinate plane can be found using two approaches.

1) Let $y = f(x)$ be the function that defines the top boundary of the region, and $y = g(x)$ be the function that defines the bottom boundary ($f(x) \geq g(x)$). Then the area of the region located between $f(x)$ and $g(x)$ from $x = a$ to $x = b$ is given by



$$A = \int_a^b \big(f(x) - g(x)\big)\,dx.$$

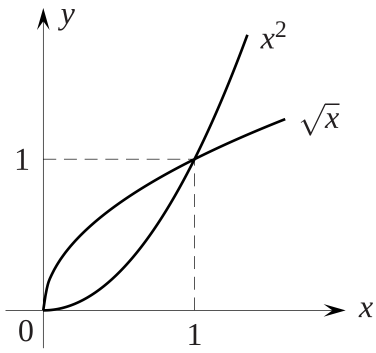
2) Let $x = \phi(y)$ be the function that defines the right boundary of the region, and let $x = \psi(y)$ be the function that defines the left boundary ($\phi(y) \geq \psi(y)$). Then the area of the region located between $\psi(y)$ and $\phi(y)$ from $y = c$ to $y = d$ is given by



$$A = \int_c^d \big(\phi(y) - \psi(y)\big)\,dy.$$

Example 11.8. Find the area of the region bounded by the curves $y = \sqrt{x}$ and $y = x^2$.

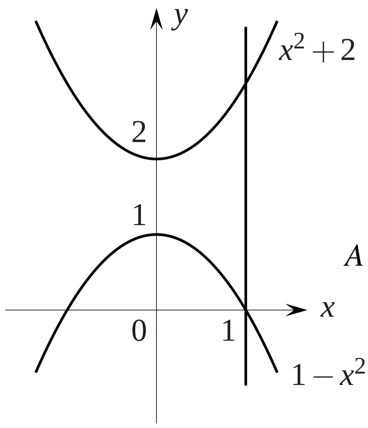
Solution. The points of intersection are the solutions of the equation $\sqrt{x} = x^2$ or $x = x^4$, so the points of intersection are $x_1 = 0$ and $x_2 = 1$. The area of the region can be found:



by the first method:
$$A = \int_0^1 \left(\sqrt{x} - x^2\right) dx = \frac{2}{3}x^{3/2}\Big|_0^1 - \frac{x^3}{3}\Big|_0^1 = \frac{2}{3} - \frac{1}{3} = \frac{1}{3}.$$
by the second method:
$$A = \int_0^1 \left(\sqrt{y} - y^2\right) dy = \frac{2}{3}y^{3/2}\Big|_0^1 - \frac{y^3}{3}\Big|_0^1 = \frac{2}{3} - \frac{1}{3} = \frac{1}{3}.$$

Example 11.9. Find the area of the region bounded by the curves $y = 1 - x^2$ and $y = x^2 + 2$ and by the lines $x = 0$ and $x = 1$.

Solution. The second method in this case is obviously much more difficult. Using the first method gives

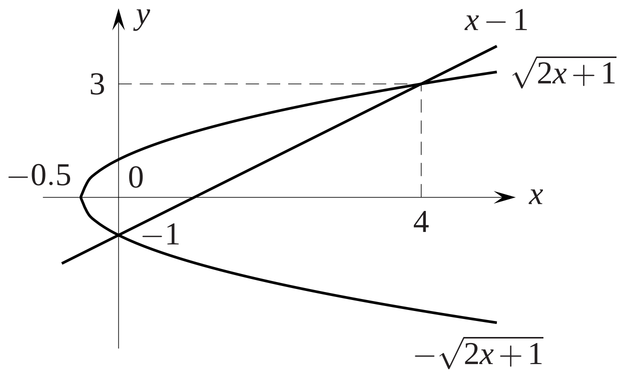


$$A = \int_0^1 \left((x^2+2) - (1-x^2)\right) dx = 2\frac{x^3}{3}\Big|_0^1 + x\Big|_0^1 = \frac{2}{3} + 1 = \frac{5}{3}.$$

Example 11.10. Find the area of the region bounded by the curves $y^2 = 2x + 1$ and $x - y - 1 = 0$.

Solution. The curve $y^2 = 2x + 1$ consists of two curves: $y = \sqrt{2x+1}$ and $y = -\sqrt{2x+1}$, while the second curve can be written as $y = x - 1$. Points of intersection:

$$\sqrt{2x+1} = x - 1 \quad \Rightarrow \quad x^2 - 4x = 0,\ x - 1 \geq 0 \quad \Rightarrow \quad x = 4;$$
$$-\sqrt{2x+1} = x - 1 \quad \Rightarrow \quad x^2 - 4x = 0,\ x - 1 \leq 0 \quad \Rightarrow \quad x = 0.$$



The area of the region can be found:

by the first method:

$$A = \int_{-1/2}^{0} \left(\sqrt{2x+1} - (-\sqrt{2x+1})\right) dx + \int_{0}^{4} \left(\sqrt{2x+1} - (x-1)\right) dx =$$

$$= 2\frac{1}{3}(2x+1)^{3/2}\Big|_{-1/2}^{0} + \frac{1}{3}(2x+1)^{3/2}\Big|_{0}^{4} - \frac{x^2}{2}\Big|_{0}^{4} + x\Big|_{0}^{4} =$$

$$= \frac{2}{3} - 0 + \frac{1}{3}(27-1) - \frac{1}{2}(16-0) + 4 - 0 = \frac{16}{3}.$$

by the second method:

$$A = \int_{-1}^{3} \left((y+1) - \frac{y^2-1}{2}\right) dy = \frac{y^2}{2}\Big|_{-1}^{3} + \frac{3}{2}y\Big|_{-1}^{3} - \frac{1}{6}y^3\Big|_{-1}^{3} =$$

$$= \frac{1}{2}(9-1) + \frac{3}{2}(3-(-1)) - \frac{1}{6}(27+1) = 4 + 6 - \frac{14}{3} = \frac{16}{3}.$$

Find the area of the region bounded by the given curves.

11.58. $y = x^2 + 1$; $x = -1$; $x = 2$; the x-axis.

11.59. $x - 2y + 4 = 0$; $x + y - 5 = 0$; $y = 0$.

11.60. $y = x^2$; $y = -2$; $x = 2$; $x = 4$.

11.61. $y = e^x + e^{-x}$; $x = 0$; $x = 3$; the x-axis.

11.62. $y = -x^2 - 1$; $y = 0$; $x = -2$; $x = 1$.

11.63. $y^2 = 4x$; $x = 1$; $x = 9$.

11.64. $3y = x^2$; $y = x$.

11.65. $y = x^3$; $y = \frac{1}{x}$; $y = 0$; $x = 4$.

11.66. $xy = 1$; $xy = 2$; $x = 1$; $x = 3$.

11.67. $y = \frac{12}{1+x^2}$; $y = x^2$.

11.68. $y = \sin x$; $y = \frac{2x}{\pi}$.

11.69. $y = 2x^2$; $y = x^2 + 4$.

11.70. $y = \frac{x}{\sqrt{1-x^2}}$; the x-axis; $x = -\sqrt{2}/2$; $x = \sqrt{2}/2$.

11.71. $x = y^2$; $y = x - 6$.

11.72. $y = \ln x$; $y = \ln x^2$; $x = 5$.

11.73. $y = e^x$; $y = e^{-x}$; $x = 1$.

11.74. $y = \sin^{-1} x$; $y = \cos^{-1} x$; the x-axis.

11.75. $y = \tan x$; the x-axis; $x = \pi/4$.

11.76. $y = \tan x$; $y = \sqrt{3}$; the y-axis.

11.77. $y = x \ln x$; $x = \frac{1}{e}$; $x = e$; the x-axis.

11.78. Let T be the region bounded by the graph of the function $y = \sin x$, the lines $x = \frac{\pi}{6}$ and $x = \frac{2\pi}{3}$, and the x-axis.

a) Find the area of T.

b) Let $x = k$ be the line that divides the region T into two parts with equal areas. Find the value of k.

11.4 Volume: Solids of revolution

The volume of a solid of revolution can be found using one of two methods: the disc and washer method and the shell method.

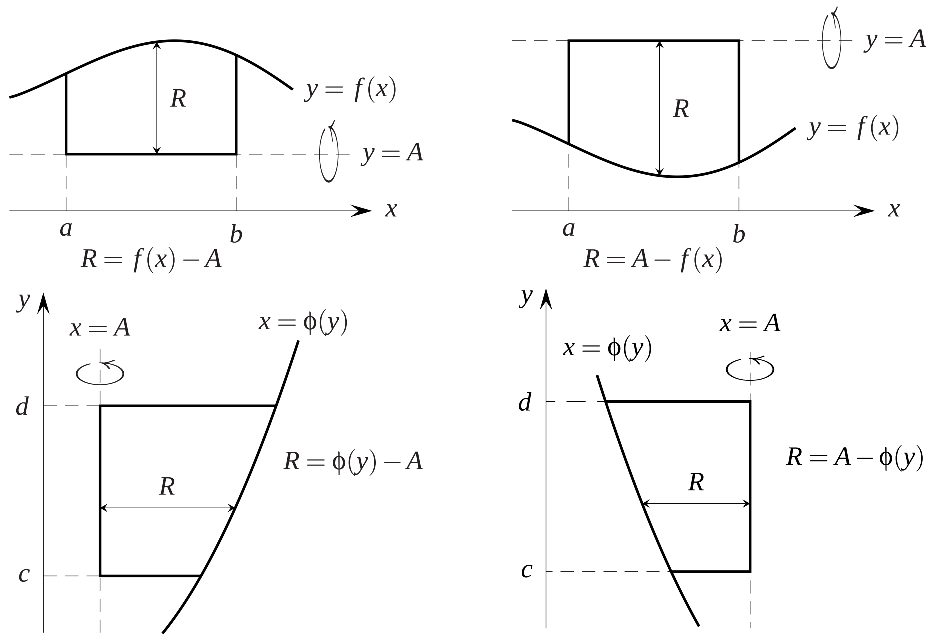
The disc and washer methods

When using this method, it is necessary to integrate along the axis of revolution. If the region is revolved about a horizontal line, integrate by x, and if the region is revolved about a vertical line, integrate by y.

The process of integration "divides"the region into a series of rectangles which are then revolved about the axis. This rotation will produce either a disc (if one of the sides of the rectangle lies on the axis) or a washer. Thus, the disc method is used if the axis of revolution is part of the boundary of the region, and the washer method is used if the axis of revolution is not part of the boundary of the region.

The disc method

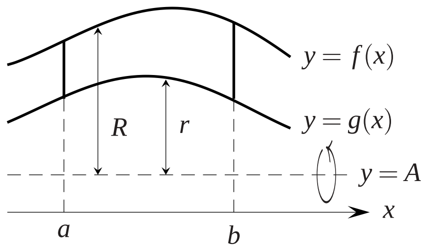
Let R be the radius of the disc, where the radius is the distance from the outer edge of the disc to the axis of revolution, as shown below:



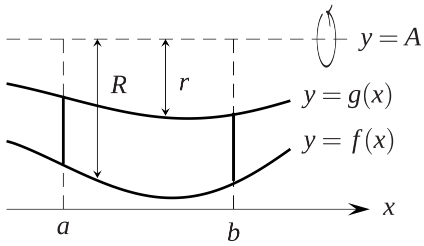
The volume is $V = \pi \int_a^b R^2 dx$ if integrating by x, and $V = \pi \int_c^d R^2 dy$ if integrating by y.

The washer method

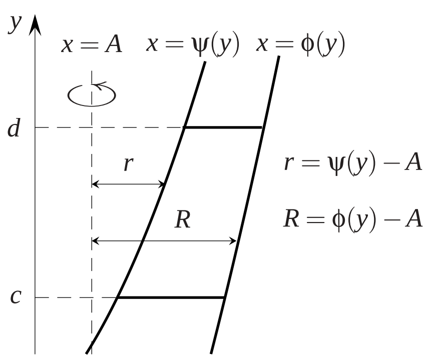
Let R and r be the outer and inner radius of the washer, respectively. The inner radius is the distance from the axis of revolution to the edge of the region closest to the axis of revolution, and the outer radius is the distance from the axis of revolution to the edge of the region furthest from the axis or revolution, as shown below:

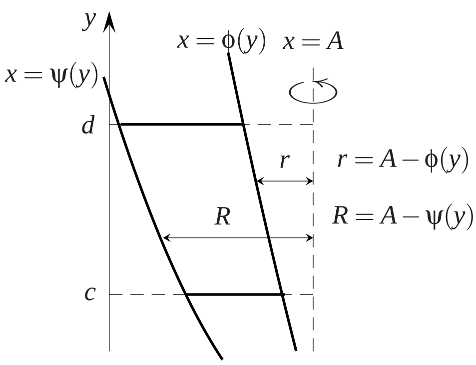


$R = f(x) - A,\ r = g(x) - A$



$R = A - f(x),\ r = A - g(x)$



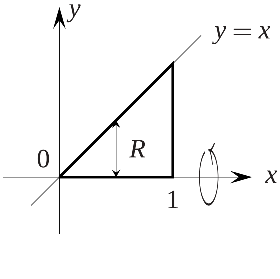


The volume is $V = \pi \int_a^b \left(R^2 - r^2\right) dx$ if integrating by x, and $V = \pi \int_c^d \left(R^2 - r^2\right) dy$ if integrating by y.

Example 11.11. Use the disc or washer method to find the volume of the solid generated when the region bounded by the lines $y = 0$, $x = 1$ and $y = x$ is revolved about a) the x-axis; b) the line $y = -1$; c) the line $y = 3$.

Solution. In all three cases the region is revolved about a horizontal line, and therefore when finding the volume of the solid using the disc or washer method it is necessary to integrate by x. In this case the range of x covered by the region is $[0, 1]$, and so integration should be done over this interval. All that is left is to determine the radius R for each case.

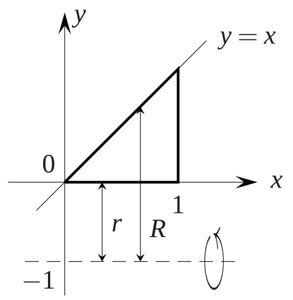
a)



The axis of rotation is part of the boundary of the region, so the disk method should be used. R is the distance between the line $y = x$ and the line $y = 0$, so $R = x - 0 = x$. The volume is

$$V = \pi \int_0^1 x^2 dx = \pi \left. \frac{x^3}{3} \right|_0^1 = \frac{\pi}{3}.$$

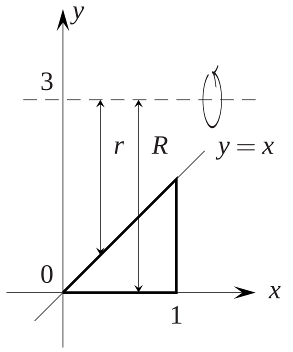
b)



In this case the axis of rotation is not part of the boundary of the region, so the washer method should be used. The outer radius R is the distance between the axis of rotation ($y=-1$) and the boundary of the region that is furthest from the axis, $y=x$; $R=x-(-1)=x+1$. The inner radius r is the distance between the axis of rotation ($y=-1$) and the boundary of the region that is closest to the axis, $y=0$. Therefore, $r=0-(-1)=1$. The volume is

$$V=\pi\int_0^1\left((x+1)^2-1^2\right)dx=\pi\int_0^1\left(x^2+2x\right)dx==\pi\left(\frac{x^3}{3}+x^2\right)\Big|_0^1=\frac{4\pi}{3}.$$

c)



As in b), the washer method should be used. The outer radius R is the distance between $y=3$ and $y=0$, so $R=3-0=3$; the inner radius is the distance between $y=3$ and $y=x$, so $r=3-x$. The volume is

$$V=\pi\int_0^1\left(3^2-(3-x)^2\right)dx=\pi\int_0^1\left(6x-x^2\right)dx=$$

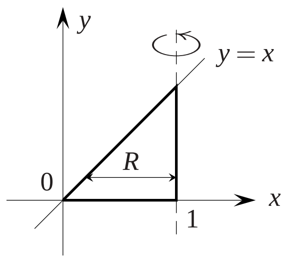
$$=\pi\left(3x^2-\frac{x^3}{3}\right)\Big|_0^1=\frac{8\pi}{3}.$$

Example 11.12. Use the disc or washer method to find the volume of the solid generated when the region bounded by the lines $y=0$, $x=1$ and $y=x$ is revolved about a) the line $x=1$; b) the y-axis; c) the line $x=4$.

Solution. In all three cases the region is revolved about a vertical line, and therefore when finding the volume of the solid using the disc or washer method it is necessary to integrate by y. In this case the range of y covered by the region is $[0,1]$, and so integration should be done over this interval. All that is left is to determine the radius R for each case.

It should be noted that since the variable of integration is y, the functions should all be expressed not in terms of x (i.e. $y=f(x)$), but in terms of y (i.e. $x=g(y)$).

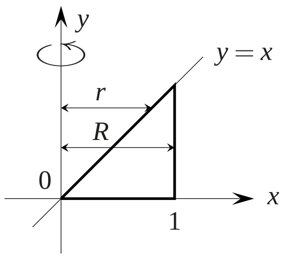
a)



The axis of rotation is part of the boundary of the region, so the disk method should be used. R is the distance between the line $y=x$ and the line $x=1$, so $R=1-y$. The volume is

$$V=\pi\int_0^1(1-y)^2dy=-\pi\left.\frac{(1-y)^3}{3}\right|_0^1=\frac{\pi}{3}.$$

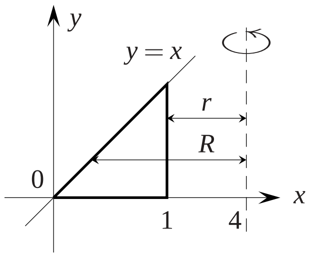
b)



In this case the axis of rotation is not part of the boundary of the region, so the washer method should be used. The outer radius R is the distance between the axis of rotation (the y-axis, $x=0$) and the boundary of the region that is furthest from the axis, $x=1$; $R=1-0=1$. The inner radius r is the distance between the axis of rotation ($x=0$) and the boundary of the region that is closest to the axis, $x=y$. Therefore, $r=y-0=y$. The volume is

$$V=\pi\int_0^1\left(1^2-y^2\right)dy=\pi\left(y-\frac{y^3}{3}\right)\Bigg|_0^1=\frac{2\pi}{3}.$$

c)



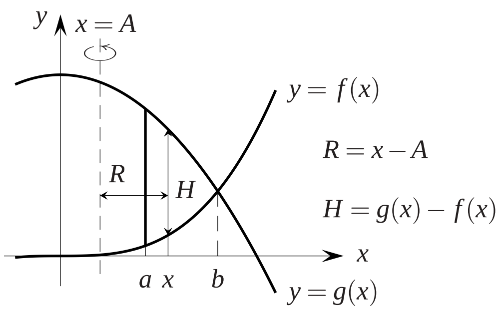
As in b), the washer method should be used. The outer radius R is the distance between $x=4$ and $y=x$, so $R=4-y$; the inner radius is the distance between $x=4$ and $x=1$, so $r=4-1=3$. The volume is

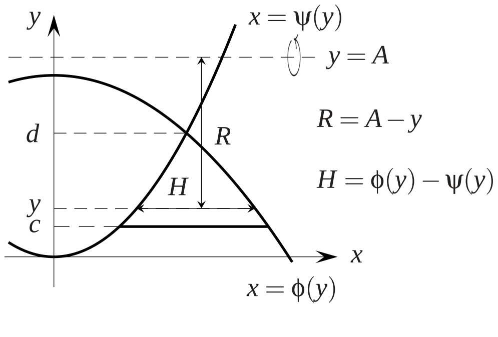
$$V=\pi\int_0^1\left((4-y)^2-3^2\right)dy=\pi\int_0^1\left(7-8y+y^2\right)dy=\pi\left(7y-4y^2+\frac{y^3}{3}\right)\Bigg|_0^1=\frac{10\pi}{3}.$$

The shell method

When using this method, it is necessary to integrate perpendicular to the axis of revolution (unlike the disc or washer method). If the region is revolved about a horizontal line, integrate by y, and if the region is revolved about a vertical line, integrate by x.

As always, the radius is the distance to the axis or revolution. For every radius R is it necessary to find the corresponding height of the shell H. The values of R and H need to be expressed in terms of the variable of integration.





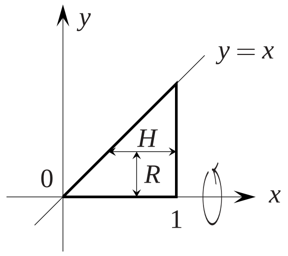
The volume will equal $V=2\pi\int_a^b RH\,dx$ if integrating by x and

$V = 2\pi \int_c^d RH dy$ if integrating by y.

Example 11.13. Use the shell method to find the volume of the solid generated when the region bounded by the lines $y = 0$, $x = 1$ and $y = x$ is revolved about a) the x-axis; b) the line $y = -1$; c) the line $y = 3$.

Solution. In this example the axis or revolution in each case is a horizontal line, and therefore it will be necessary to integrate by y.

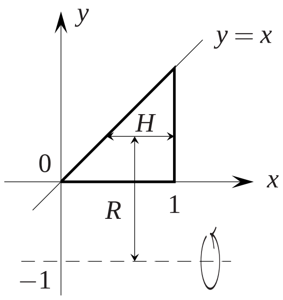
a)



The radius is the distance between the axis of revolution and the current value of y: $R = y - 0 = y$. The height of the shell is the horizontal distance between the line $y = x$ and the line $x = 1$, or $H = 1 - y$. Therefore, the volume is

$$V = 2\pi \int_0^1 y(1-y)dy = 2\pi \left(\frac{y^2}{2} - \frac{y^3}{3}\right)\Bigg|_0^1 = \frac{2\pi}{6} = \frac{\pi}{3}.$$

b)

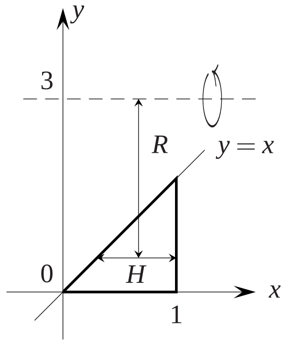


The radius is as before the distance between the axis of revolution $y = -1$ and the current value of y: $R = y - (-1) = y + 1$. The height of the shell will be $H = 1 - y$, exactly as in a). The volume is therefore

$$V = 2\pi \int_0^1 (y+1)(1-y)dy = 2\pi \int_0^1 \left(1 - y^2\right) dy =$$

$$= 2\pi \left(y - \frac{y^3}{3}\right)\Bigg|_0^1 = \frac{4\pi}{3}.$$

c)



The radius, found in the same way as in a) and b), is $R = 3 - y$, while the height again remains the same: $H = 1 - y$. The volume is

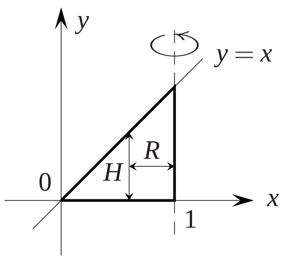
$$V = 2\pi \int_0^1 (3-y)(1-y)dy = 2\pi \int_0^1 (3 - 4y + y^2)dy =$$

$$= 2\pi \left(3y - 2y^2 + \frac{y^3}{3}\right)\Bigg|_0^1 = \frac{8\pi}{3}.$$

Example 11.14. Use the shell method to find the volume of the solid generated when the region bounded by the lines $x = 0$, $x = 1$ and $y = x$ is revolved about a) the line $x = 1$; b) the y-axis; c) the line $x = 4$.

Solution. In all three cases the region is revolved about a vertical line, and therefore when finding the volume of the solid using the shell method it is necessary to integrate by x. In this case the range of x covered by the region is $[0,1]$, and so integration should be done over this interval. All that is left is to determine the radius R for each case.

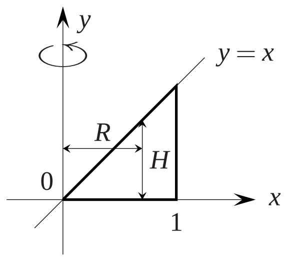
a)



The radius is the distance between the axis of revolution $x=1$ and the current value of x: $R=1-x$. The height of the shell is the vertical distance between the line $y=x$ and the line $y=0$: $H=x$. Therefore, the volume is

$$V = 2\pi \int_0^1 (1-x)x\,dx = 2\pi \left(\frac{x^2}{2} - \frac{x^3}{3}\right)\Bigg|_0^1 = \frac{\pi}{3}.$$

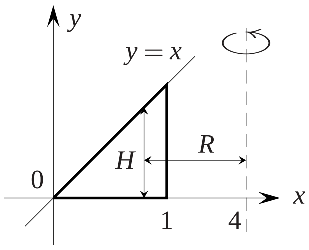
b)



The radius is as before the distance between the axis of revolution $x=0$ and the current value of x: $R=x-0=x$. The height of the shell will be $H=x$, exactly as in a). The volume is therefore

$$V = 2\pi \int_0^1 x^2 dx = 2\pi \left.\frac{x^3}{3}\right|_0^1 = \frac{2\pi}{3}.$$

c)



The radius, found in the same way as in a) and b), is $R=4-x$, while the height again remains the same: $H=x$. The volume is

$$V = 2\pi \int_0^1 x(4-x)dx = 2\pi \left(2x^2 - \frac{x^3}{3}\right)\Bigg|_0^1 = \frac{10\pi}{3}.$$

11.79. Find the volume of the solid generated when the region bounded by the curves $y=x$, $x=3$, $x=7$ and $y=0$ is revolved about the x-axis.

11.80. Find the volume of the solid generated when the region bounded by the curves $y=3x$, $y=2$, $y=4$ and $x=0$ is revolved about the y-axis.

11.81. Find the volume of the solid generated when the region bounded by the curves $y^2=4x$, and $y=x$ is revolved about the x-axis.

11.82. Find the volume of the solid generated when the region bounded by the curves $y^2=4x$ and $y=x$ is revolved about the line $x=-1$.

11.83. Find the volume of the solid generated when the region bounded by the curves $y^2=4x$ and $y=x$ is revolved about the line $x=4$.

11.84. Find the volume of the solid generated when the region bounded by the curves $y=x^2$ and $y^2=8x$ is revolved about a) the x-axis; b) the y-axis.

11.85. Find the volume of the solid generated when the region bounded by the curves $y=x^2$ and $y^2=x$ is revolved about the x-axis.

11.86. Find the volume of the solid generated when the region bounded by the curves $y=x-x^2$ and $y=0$ is revolved about a) the x-axis; b) the y-axis; c) the line $x=2$; d) the line $x=-2$; e) the line $y=-1$; f) the line $y=2$.

11.87. Find the volume of the solid generated when the region bounded by the curve $x^2 - y^2 = 1$ and the lines $y = 0$, $x = 1$ and $x = 2$ is revolved about the y-axis.

11.88. Find the volume of the solid generated when the region in the first quadrant bounded by the curve $y^2 - x + 1 = 0$ and the lines $x = 2$, $y = 0$ is revolved about a) the x-axis; b) the y-axis.

11.5 Volume: Solids with known cross-sections

Solids with known cross-sections are described by the form of their base and the form of the cross-sections perpendicular to that base. If the cross sections are perpendicular to the x-axis, then integrate by x; if the cross sections are perpendicular to the y-axis, then integrate by y. The volume will be $V = \int\limits_a^b S(x)dx$ or $V = \int\limits_c^d S(y)dy$, depending on what variable is being integrated.

11.89. The base of a solid is the region in the first quadrant bounded by the x-axis, the y-axis and the line $x + 2y = 4$. If cross section of the solid perpendicular to the x-axis are semicircles, what is the volume of the solid?

11.90. The base of a solid B is the region enclosed by the graph of $y = e^x$, the line $y = e$, and the y-axis. If the cross sections of B perpendicular to the y-axis are squares, find the volume of B.

11.91. Find the volume of the solid with a base defined by the lines $y = x - 1$, $y = 2 - x$, and the y-axis, and the cross-section parallel to the y-axis of which are squares.

11.92. A solid has a circular base of radius 1 with the center at the origin. Parallel cross sections perpendicular to the base are equilateral triangles. Find the volume of the solid.

11.93. The base of a solid is the circle $x^2 + y^2 = 16$, and every plane section perpendicular to the x-axis is a rectangle whose height is twice the distance of the plane section from the origin. Find the volume of the solid.

11.6 Position, Velocity and Acceleration

Given the velocity function $v(t)$ of an object and the position of this object at some moment $s(t_0) = s_0$, the position of the object for any value of t is given by

$$s(t) = \int\limits_{t_0}^{t} v(u)du + s_0.$$

In the same way, given the acceleration function $a(t)$ and an initial condition for velocity $v(t_0) = v_0$,

$$v(t) = \int_{t_0}^{t} a(u)du + v_0.$$

The total distance that an object travels from time $t = t_1$ to $t = t_2$ is given by

$$\int_{t_1}^{t_2} |v(t)|\, dt.$$

The displacement is given by

$$s(t_2) - s(t_1) = \int_{t_1}^{t_2} v(u)du.$$

Example 11.15. The acceleration of a particle moving along the x-axis is given by $a(t) = t^2 + 3t$. Find the position of the object at any time t if the particle's velocity at $t = 1$ was 2, and the particle was located at $s = 2$ at time $t = 2$.

Solution. First find the velocity function:

$$v(t) = \int_1^t \left(u^2 + 3u\right) du + 2 = \frac{u^3}{3}\bigg|_1^t + \frac{3}{2}u^2\bigg|_1^t + 2 = \frac{t^3}{3} - \frac{1}{3} + \frac{3}{2}(t^2 - 1) + 2 =$$

$$= \frac{1}{3}t^3 + \frac{3}{2}t^2 + \frac{1}{6}.$$

Now find the position of the particle:

$$s(t) = \int_2^t \left(\frac{1}{3}u^3 + \frac{3}{2}u^2 + \frac{1}{6}\right) du + 2 = \frac{1}{12}u^4\bigg|_2^t + \frac{1}{2}u^3\bigg|_2^t + \frac{1}{6}u\bigg|_2^t + 2 =$$

$$= \frac{1}{12}(t^4 - 16) + \frac{1}{2}(t^3 - 8) + \frac{1}{6}(t - 2) + 2 = \frac{1}{12}t^4 + \frac{1}{2}t^3 + \frac{1}{6}t - 1.$$

11.94. The position of a particle, which starts moving at time $t = 0$, is given by $x(t) = 6t - t^2$. Where will the particle be at time $t = 5$? What distance will the particle have traveled?

11.95. The speed of a point moving along a straight line, in meters per second, is given by $v(t) = 3t^2 - 3t$. How far away will the point be from its starting place in 4 seconds? What is the total distance the point will have traveled?

11.96. An automobile traveling at a speed of 48 kilometers per hour begins to brake and stops in 3 seconds. Find the distance the automobile had traveled before stopping if it was decelerating at a constant rate.

11.97. A jet accelerates from 100 meters per second to 200 meters per second in 20 seconds. If its acceleration is constant, find the acceleration and the distance it travels during this time.

11.98. A bicyclist traveling at 37 ft/sec ceases pedaling and coasts until his speed decreases to 29ft/sec. The deceleration is proportionate to the square of the velocity, where the constant of proportionality is k. a) Find the velocity of the bicyclist as a function of k and time t. b) Find k if the deceleration is 1 ft/sec^2 at 37 ft/sec. c) How long will it take for the bicycle to reach 29 ft/sec? d) How far will the bicycle have traveled by the time it reaches the speed of 29 ft/sec?

11.99. The acceleration of a particle moving along the x-axis at time t is given by $a(t) = 6t - 2$. If the velocity is 25 when $t = 3$ and the position is 10 when $t = 1$, then find the position $x(t)$ as a function t.

11.100. At time t $(t > 0)$, the acceleration of a particle moving on the x-axis is $a(t) = t + \sin t$. At time $t = 0$, the velocity of the particle is -2. For what value of t will the velocity of the particle be zero?

11.101. A ball is thrown upward from ground level with an initial speed of 35 meters per second so that its height is given by $y = 35t - 5t^2$. How high does the ball go? How fast will it strike the ground?

11.102. When fired from rest at ground level, a small rocket rises vertically so that its acceleration after t seconds is $6t \frac{\text{m}}{\text{sec}^2}$. This continues for the 10 seconds that its fuel lasts. Thereafter, the rocket's acceleration is $10 \frac{\text{m}}{\text{sec}^2}$ downward, due to gravity. In how many seconds will the rocket strike the ground after falling back?

Chapter 12.
IMPROPER INTEGRALS

12.1 Unbounded functions (Type I)

Definition. Let f be continuous on $[a,b)$ and discontinuous at $x=b$. Then

$$\int_a^b f(x)dx = \lim_{c\to b^-} \int_a^c f(x)dx,$$

provided the limit exists.

Definition. Let f be continuous on $(a,b]$ and discontinuous at $x=a$. Then

$$\int_a^b f(x)dx = \lim_{c\to a^+} \int_c^b f(x)dx,$$

provided the limit exists.

Definition. Let f be continuous for all $x \in [a,b]$ except for $x=t$, $t \in (a,b)$. Assume further that $\int_a^t f(x)dx$ and $\int_t^b f(x)dx$ exist as given above. Then

$$\int_a^b f(x)dx = \int_a^t f(x)dx + \int_t^b f(x)dx.$$

Note that, in order for $\int_a^b f(x)dx$ to exist when f is unbounded at $x_0 \in (a,b)$, it is necessary for *both* of the integrals $\int_a^t f(x)dx$ and $\int_t^b f(x)dx$ to exist.

Example 12.1. Find the values of p such that the integral $\int_0^1 \frac{dx}{x^p}$ is convergent.

Solution. We will first consider the case $p=1$. We have:

$$\int_0^1 \frac{dx}{x} = \lim_{t\to 0^+} \ln x\Big|_t^1 = -\lim_{t\to 0^+} \ln t \quad \nexists.$$

Assuming now $p \neq 1$, we have:

$$\int_0^1 \frac{dx}{x^p} = \lim_{t\to 0^+} \int_t^1 \frac{dx}{x^p} = \lim_{t\to 0^+} \left. \frac{x^{-p+1}}{-p+1} \right|_t^1 = \frac{1}{1-p} - \lim_{t\to 0^+} \frac{t^{1-p}}{1-p}.$$

If $1-p>0$, then this limit exists; if $1-p<0$, then it does not (remember that the case $p=1$ was considered separately). Thus, the integral $\int_0^1 \frac{dx}{x^p}$ exists for $p<1$.

Example 12.2. Determine whether $\int_0^\pi \frac{dx}{\cos^2 x}$ converges.

Solution. The integrand is unbounded at $x=\pi/2$. Therefore,

$$\int_0^{\pi/2} \frac{dx}{\cos^2 x} = \int_0^{\pi} \frac{dx}{\cos^2 x} + \int_{\pi/2}^{\pi} \frac{dx}{\cos^2 x}.$$

We consider the first integral:

$$\int_0^{\pi} \frac{dx}{\cos^2 x} = \lim_{t\to\pi/2^-} \tan x\big|_0^t = \lim_{t\to\pi/2^-} \tan t = \infty.$$

Without even considering $\int_{\pi/2}^{\pi} \frac{dx}{\cos^2 x}$, we conclude that $\int_0^{\pi} \frac{dx}{\cos^2 x}$ does not converge.

Find the following improper integrals, or determine that they do not converge.

12.1. $\int_1^{33} \frac{dx}{\sqrt[5]{x-1}}$

12.2. $\int_0^{3} \frac{dx}{\sqrt[5]{x-1}}$

12.3. $\int_0^{3} \frac{dx}{\sqrt[5]{(x-1)^6}}$

12.4. $\int_{-1}^{1} \sqrt{\frac{1+x}{1-x}}dx$

12.5. $\int_0^{\pi/2} \frac{dx}{1+\tan x}$

12.6. $\int_0^{1} \frac{dx}{\sqrt{1-x^2}}$

12.7. $\int_1^{9} \frac{dx}{\sqrt[3]{x-1}}$

12.8. $\int_0^{1} \frac{dx}{x+\sqrt{x}}$

12.9. $\int_0^{1} \frac{dx}{e^x-1}$

12.10. $\int_0^{1} (1-x)^{-2/3}dx$

12.11. $\int_0^{\pi/2} \tan x dx$

12.12. $\int_0^1 \sqrt{x} \ln x dx$

12.13. $\int_0^{\infty} \frac{dx}{\sqrt{x}(1+x)}$

12.14. $\int_0^{\pi/2} \frac{dx}{1-\tan^2 x}$

12.2 Unbounded intervals (Type II)

Definition. If $\int_a^b f(x)dx$ exists for all $b > a$, then

$$\int_a^{\infty} f(x)dx = \lim_{b\to+\infty} \int_a^b f(x)dx,$$

provided the limit exists.

Definition. If $\int_a^b f(x)dx$ exists for all $a < b$, then

$$\int_{-\infty}^{b} f(x)dx = \lim_{a\to-\infty} \int_a^b f(x)dx,$$

provided the limit exists.

Since these improper integrals are in fact limits at infinity just like in section 3.1, the concept of *convergence* is applicable. Thus, if e.g. $\int_a^{\infty} f(x)dx$ exists, we may also say that it *converges,* and if $\int_a^{\infty} f(x)dx$ does not exist, we may also say that it *diverges.*

There is a close link between improper integrals on the interval $[a,\infty)$ and infinite series. Indeed, taking sequentially $b = 2,3,4,\ldots$ the integrals $\int_1^2 f(x)dx$, $\int_1^3 f(x)dx$, $\int_1^4 f(x)dx$, etc. form a sequence very much like the N-th partial sums in series. Many properties of improper integrals on unbounded intervals may be analyzed by using series, and vice versa.

Theorem (Necessary condition for the convergence of an improper integral on $[a,\infty)$)

If $\int_a^{\infty} f(x)dx$ converges, then $\lim_{x\to\infty} f(x) = 0$.

Definition. Assuming that $\int\limits_{-\infty}^{c} f(x)dx$ and $\int\limits_{c}^{\infty} f(x)dx$ are both convergent for any value of c, then

$$\int\limits_{-\infty}^{\infty} f(x)dx = \int\limits_{-\infty}^{c} f(x)dx + \int\limits_{c}^{\infty} f(x)dx.$$

Example 12.3. Find the following improper integrals, or determine that they do not converge.

$$\text{a)} \int\limits_{1}^{\infty} \frac{dx}{x}; \quad \text{b)} \int\limits_{1}^{\infty} \frac{dx}{x^2}; \quad \text{c)} \int\limits_{-\infty}^{\infty} xe^{-x^2} dx.$$

Solution. a) $\int\limits_{1}^{\infty} \frac{dx}{x} = \lim\limits_{b\to+\infty} \int\limits_{1}^{b} \frac{dx}{x} = \lim\limits_{b\to+\infty} \ln b = \infty$. Therefore, this integral diverges.

b) $\int\limits_{1}^{\infty} \frac{dx}{x^2} = \lim\limits_{b\to+\infty} \int\limits_{1}^{b} \frac{dx}{x^2} = \lim\limits_{b\to+\infty} \left(-\frac{1}{x}\right)\Bigg|_1^b = \lim\limits_{b\to+\infty} \left(1 - \frac{1}{b}\right) = 1$. This integral converges.

c) We can split this integral at any point. Choose $c = 0$ for simplicity, then

$$\int\limits_{-\infty}^{\infty} xe^{-x^2} dx = \int\limits_{-\infty}^{0} xe^{-x^2} dx + \int\limits_{0}^{\infty} xe^{-x^2} dx.$$

We have:

$$\int\limits_{-\infty}^{0} xe^{-x^2} dx = \lim_{a\to-\infty} \int\limits_{a}^{0} xe^{-x^2} dx = \lim_{a\to-\infty} \left(-\frac{1}{2}e^{-x^2}\right)\Bigg|_a^0 = \lim_{a\to-\infty} \left(-\frac{1}{2} + \frac{1}{2}e^{-a^2}\right) = -\frac{1}{2},$$

and

$$\int\limits_{0}^{\infty} xe^{-x^2} dx = \lim_{b\to\infty} \int\limits_{0}^{b} xe^{-x^2} dx = \lim_{b\to\infty} \left(-\frac{1}{2}e^{-x^2}\right)\Bigg|_0^b = \lim_{b\to\infty} \left(-\frac{1}{2}e^{-b^2} + \frac{1}{2}\right) = \frac{1}{2}.$$

Since both of these integrals are convergent, the initial integral is also convergent, and it equals

$$\int\limits_{-\infty}^{\infty} xe^{-x^2} dx = \int\limits_{-\infty}^{0} xe^{-x^2} dx + \int\limits_{0}^{\infty} xe^{-x^2} dx = -\frac{1}{2} + \frac{1}{2} = 0.$$

One may have said that the integral equals zero, simply because the integrand is an odd function and the interval of integration is symmetrical with respect to the origin; however, this argument ignores the existence issue and is thus invalid.

Find the following improper integrals, or determine that they do not converge.

12.15. $\int_{-\infty}^{0} \frac{dx}{\sqrt{3-x}}$

12.16. $\int_{-2}^{\infty} \sin x dx$

12.17. $\int_{1}^{\infty} \frac{dx}{x^5}$

12.18. $\int_{2}^{\infty} \frac{dx}{(3+x)^3}$

12.19. $\int_{-1}^{\infty} e^{-5x} dx$

12.20. $\int_{1}^{\infty} \frac{xdx}{(x^2+5)^3}$

12.21. $\int_{0}^{\infty} \frac{x^2}{x^3+1} dx$

12.22. $\int_{-\infty}^{\infty} \frac{dx}{25+4x^2}$

12.23. $\int_{3}^{\infty} \frac{e^{-3x} dx}{9+e^{-3x}}$

12.24. $\int_{-\infty}^{2} \frac{dx}{(x-5)^{4/3}}$

12.25. $\int_{1}^{\infty} (7x+4)^{-4/5} dx$

12.26. $\int_{-\infty}^{-5} e^{9x} dx$

12.27. $\int_{2}^{\infty} \frac{\ln x}{x} dx$

12.28. $\int_{0}^{\infty} e^{-x} \cos x dx$

12.29. $\int_{-2}^{\infty} x^2 e^{-2x} dx$

12.30. $\int_{0}^{\infty} \frac{e^x - 1}{e^{2x}+1} dx$

12.31. $\int_{0}^{\infty} \frac{dx}{x^3+8}$

12.32. $\int_{0}^{\infty} \frac{dx}{x^2+2x+2}$

12.33. The unbounded region enclosed by the Gaussian curve $y = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$ and the x-axis is revolved about the x-axis. Given that the area under the Gaussian curve is equal to 1, what is the volume of the corresponding solid of revolution?

12.34. The region enclosed by the Gaussian curve $y = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$ $(x \geq 0)$ and the x-axis is revolved about the y-axis. Find the volume of the corresponding solid of revolution.

12.35. (Gabriel's Horn) Consider the unbounded region enclosed by the x-axis, the line $x = 1$ and the curve $y = \frac{1}{x}$. Show that the area of this region is infinite, but the volume of the solid produced when the region is revolved about the x-axis if finite.

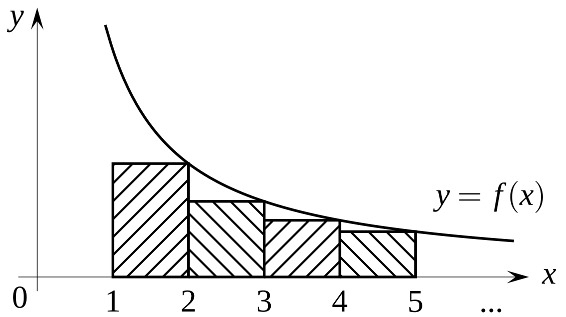
12.3 Convergence issues

Theorem (The comparison test)

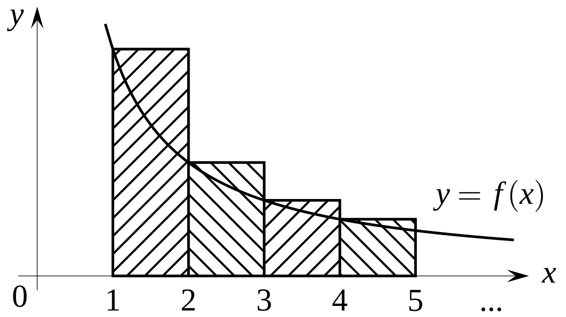
Let f and g be functions such that $0 \le f(x) \le g(x)$ for all $x \in (a,b)$, where the interval (a,b) could be infinite, or f and g may be unbounded. If $\int_a^b g(x)dx$ converges, then $\int_a^b f(x)dx$ also converges; and if $\int_a^b f(x)dx$ diverges, then $\int_a^b g(x)dx$ also diverges.

A special application of the comparison test establishes the link between improper integrals of positive functions on the interval $[a,\infty)$ and positive infinite series. Thus, the entire apparatus for examining the convergence of positive series (see section 7.2) can be used to determine the convergence of improper integrals of positive functions.

Theorem. If $\sum_{n=1}^{\infty} f(n)$ diverges, then $\int_1^{\infty} f(x)dx$ diverges as well; if $\int_1^{\infty} f(x)dx$ converges, then $\sum_{n=1}^{\infty} f(n)$ converges as well.



Theorem. If $\sum_{n=1}^{\infty} f(n)$ converges, then $\int_1^{\infty} f(x)dx$ converges as well; If $\int_1^{\infty} f(x)dx$ diverges, then $\sum_{n=1}^{\infty} f(n)$ diverges as well.



This is the substance of the proof of the criteria for the convergence of the generalized harmonic series (see page 62).

Also like series, the concepts of *absolute* and *conditional* convergence are applicable:

Definition. If the improper integral $\int_a^b |f(x)|dx$ converges, then $\int_a^b f(x)dx$ **converges absolutely**; if $\int_a^b f(x)dx$ converges but $\int_a^b |f(x)|dx$ does not, then $\int_a^b f(x)dx$ **converges conditionally**.

Determine whether or not the following integrals converge using the comparison test.

12.36. $\int_{-2}^{3} \frac{dx}{x^3}$

12.37. $\int_{0}^{\infty} e^{-7x}|\sin x|dx$

12.38. $\int_{8}^{\infty} \frac{dx}{\sqrt[3]{x}-1}$

12.39. $\int_{1}^{\infty} \frac{\sin^2(1/x)}{\sqrt{x}}dx$

12.40. $\int_{3}^{\infty} \frac{\ln x dx}{x^4+1}$

12.41. $\int_{1}^{\infty} \frac{x^3-2x^2+x+1}{x^4+x+8}dx$

12.42. $\int_{1}^{\infty} \frac{\ln x dx}{x^2+1}$

12.43. $\int_{0}^{\infty} \frac{e^{-x^2}}{x^2}dx$

12.44. $\int_{0}^{1} \frac{\sin\sqrt{x}}{x^4+x}dx$

12.45. $\int_{0}^{\pi/2} \frac{dx}{\sqrt{\sin x}}$

12.46. $\int_{0}^{\pi/2} \frac{1+\cos x}{x}dx$

12.47. $\int_{1}^{\infty} \frac{\ln x dx}{1+x^2}$

12.48. $\int_{0}^{\infty} \frac{dx}{\sqrt[3]{x}+x^2}$

12.49. $\int_{1}^{\infty} \frac{dx}{\sqrt{1+x^4}}$

12.50. $\int_{1}^{2} \frac{dx}{x^2\ln x}$

12.51. $\int_{2}^{\infty} \frac{dx}{x^2\ln x}$

12.52. $\int_{1}^{\infty} \frac{dx}{x^2\ln x}$

Determine whether or not the following integrals converge by comparing them to the appropriate series.

12.53. $\int_1^\infty \left(\frac{x+1}{3x}\right)^x dx$

12.54. $\int_1^\infty \frac{2x-1}{(\sqrt{2})^x} dx$

12.55. $\int_1^\infty \frac{x\sqrt[3]{x}}{\sqrt[4]{x^7+3}} dx$

12.56. $\int_1^\infty \frac{x^3}{4^x} dx$

12.57. $\int_1^\infty \frac{4^x dx}{x3^x}$

12.58. $\int_1^\infty \frac{dx}{3^x\sqrt{x}}$

12.59. $\int_1^\infty \left(\frac{x}{x+1}\right)^x dx$

12.60. $\int_1^\infty \left(\frac{x}{x+1}\right)^{x^2} dx$

12.61. $\int_1^\infty \frac{dx}{\ln^x(x+1)}$

12.62. $\int_1^\infty \tan\frac{3+x}{x^2} dx$

12.63. $\int_1^\infty \left(\tan^{-1}\left(\frac{1}{x}\right)\right)^x dx$

12.64. $\int_1^\infty \sin\frac{\pi}{2^x} dx$

12.65. Consider the divergent integrals $\int_1^\infty f(x)dx$ and $\int_1^\infty g(x)dx$. What can be said about the convergence of i) $\int_1^\infty (f(x)+g(x))\,dx$; ii) $\int_1^\infty f(x)g(x)dx$; iii) $\int_1^\infty \frac{f(x)}{g(x)}dx$, assuming $g(x) \neq 0$ for all $x \geq 1$?

12.66. Consider the divergent integral $\int_1^\infty f(x)dx$ and the convergent integral $\int_1^\infty g(x)dx$. What can be said about the convergence of i) $\int_1^\infty (f(x)+g(x))\,dx$; ii) $\int_1^\infty f(x)g(x)dx$; iii) $\int_1^\infty \frac{f(x)}{g(x)}dx$, assuming $g(x) \neq 0$ for all $x \geq 1$; iv) $\int_1^\infty \frac{g(x)}{f(x)}dx$, assuming $f(x) \neq 0$ for all $x \geq 1$?

12.67. If $\int_1^\infty f(x)dx$ converges and $|g(x)| < |f(x)|$ for all $x \geq 1$, is it true that the integral $\int_1^\infty g(x)dx$ converges as well?

12.68. Show that if $\sum_{n=1}^\infty a_n$ converges and $a_n \geq 0$, then $\sum_{n=1}^\infty na_{n^2}$ also converges.

12.4 The principal value of an improper integral

In many applications, it is possible to assign a value to divergent improper integrals in a logical way. This is done by introducing the concept of *principal value*.

Definition. The **Cauchy principal value** of a Type I improper integral is

$$\text{p.v.}\int_a^b f(x)dx = \lim_{\varepsilon\to 0^+}\left(\int_a^{t-\varepsilon} f(x)dx + \int_{t+\varepsilon}^b f(x)dx\right),$$

where $\lim\limits_{x\to t^-} f(x) = \pm\infty$ and $\lim\limits_{x\to t^+} f(x) = \mp\infty$.

Definition. The **Cauchy principal value** of a Type II improper integral is

$$\text{p.v.}\int_{-\infty}^{\infty} f(x)dx = \lim_{t\to\infty}\int_{-t}^{t} f(x)dx.$$

Some alternative notations for the principal value of an integral are $PV\int_a^b f(x)dx$, $P\int_a^b f(x)dx$ and $⨍_a^b f(x)dx$.

Example 12.4. Find $\text{p.v.}\int_{-2}^{3}\frac{dx}{x}$.

Solution. The integrand is unbounded at $x = 0$. We have:

$$\text{p.v.}\int_{-2}^{3}\frac{dx}{x} = \lim_{\varepsilon\to 0^+}\left(\int_{-2}^{-\varepsilon}\frac{dx}{x} + \int_{\varepsilon}^{3}\frac{dx}{x}\right) =$$

$$= \lim_{\varepsilon\to 0^+}\left(\ln|x|\Big|_{-2}^{-\varepsilon} + \ln|x|\Big|_{\varepsilon}^{3}\right) = \lim_{\varepsilon\to 0^+}(\ln\varepsilon - \ln 2 + \ln 3 - \ln\varepsilon) = \ln\frac{3}{2}.$$

Note that this result is the same as the result of the naïve (and incorrect!) calculation $\int_{-2}^{3}\frac{dx}{x} = \ln|x|\Big|_{-2}^{3} = \ln\frac{3}{2}$ (in actual fact, this integral does not exist). Cauchy's principal value, however, gives a mathematically rigorous way to interpret this naïve result.

Chapter 13.
DOUBLE AND ITERATED INTEGRALS

13.1 Description of regions on the coordinate plane

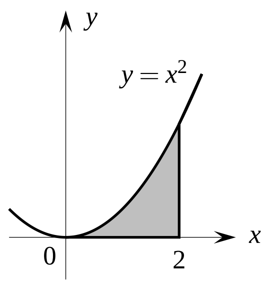
Regions on the coordinate plane are often described by the graphs of functions or equations that give its boundaries. However, it will be necessary to transform this information into systems of inequalities in order to discuss double and iterated integrals later.

We will consider two methods of describing regions on the coordinate plane with the use of inequalities. Graphically, these methods use either vertical or horizontal lines.

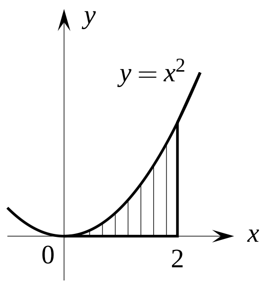
Using vertical lines, it is necessary to first determine the range of x that is covered by the region. The second step is to then determine the range of y that is covered by the region for each concrete value of x.

Using horizontal lines, the process is reversed: first it is necessary to find the range of y that is covered by the region, and then the range of x that is covered by the region for each concrete value of y.

Example 13.1. Describe the region D bounded by the lines $y = 0$, $y = x^2$ and $x = 2$ using inequalities.

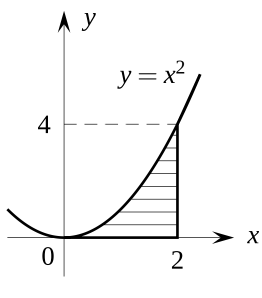


Solution. We will first use vertical lines. Note that x changes from 0 to 2.



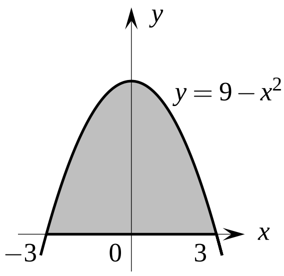
We see that each vertical line goes from the x-axis ($y = 0$) to the curve $y = x^2$. Therefore, the answer is $D = \left\{(x,y) : 0 \le x \le 2, 0 \le y \le x^2\right\}$.

We will now consider horizontal lines. Note that y changes from 0 to 4.

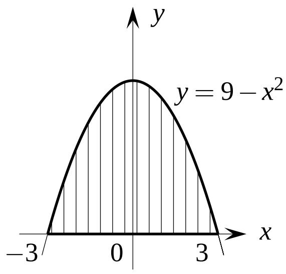


We see that each horizontal line goes from the curve $y = x^2$ ($x = \sqrt{y}$) to the vertical line $x = 2$. Therefore, the answer is $D = \left\{(x,y) : 0 \le y \le 4, \sqrt{y} \le x \le 2\right\}$.

Example 13.2. Describe the region D bounded by the line $y=0$ and the curve $y=9-x^2$ using inequalities.

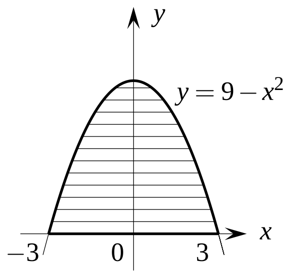


Solution. We will first use vertical lines. Note that x changes from -3 to 3.



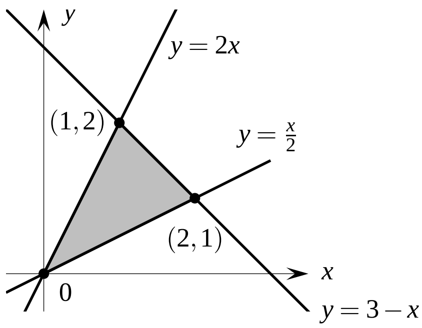
We see that each vertical line goes from the x-axis ($y=0$) to the curve $y=9-x^2$. Therefore, the answer is $D=\left\{(x,y): -3\le x\le 3, 0\le y\le 9-x^2\right\}$.

We will now consider horizontal lines. Note that y changes from 0 to 9.

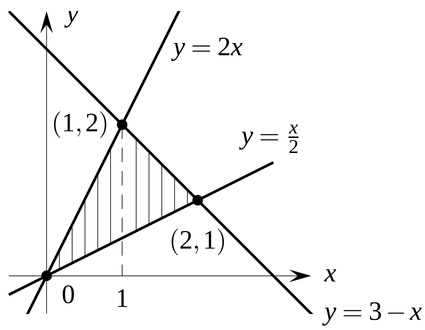


We see that each horizontal line goes from one branch of the curve $y=9-x^2$ to the other, i.e. from the curve $x=-\sqrt{9-y}$ to the curve $x=\sqrt{9-y}$. Therefore, the answer is $D=\left\{(x,y): 0\le y\le 9, -\sqrt{9-y}\le x\le\sqrt{9-y}\right\}$.

Example 13.3. Describe the region D bounded by the lines $y=\frac{x}{2}$, $y=2x$ and $y=3-x$ using inequalities.



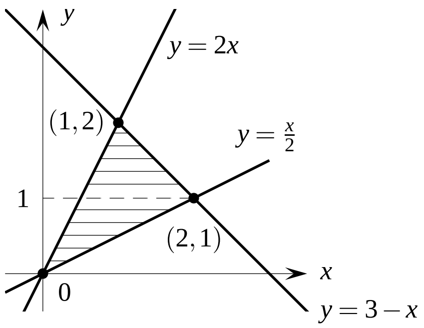
Solution. We will first use vertical lines. Note that x changes from 0 to 2.



The main difference in this problem compared to the preceding examples is that while the *bottom* boundary $y=\frac{x}{2}$ is the same for all vertical lines, the *top* boundary is not: $y=2x$ for $0\le x\le 1$ and $y=3-x$ for $1\le x\le 2$. Therefore, the answer is

$$D=\left\{(x,y): 0\le x\le 1, \tfrac{x}{2}\le y\le 2x\right\}\bigcup\left\{(x,y): 1\le x\le 2, \tfrac{x}{2}\le y\le 3-x\right\}.$$

We will now consider horizontal lines. Note that y changes from 0 to 2.



Just like for vertical lines, here it is necessary to take into account that while the *left* boundary is always the same ($y=2x$, or $x=\frac{y}{2}$), the *right* boundary is not: $y=\frac{x}{2}$ ($x=2y$) for $0\le y\le 1$, and $y=3-x$ ($x=3-y$) for $1\le y\le 2$. Therefore, the answer is

$$D=\left\{(x,y): 0\le y\le 1, \frac{y}{2}\le x\le 2y\right\}\bigcup\left\{(x,y): 1\le y\le 2, \frac{y}{2}\le x\le 3-y\right\}.$$

Use inequalities to describe the regions bounded by the graphs of the functions given below.

13.1. $y=\frac{x}{2}, \quad y=0, \quad x=4$

13.2. $y=x, \quad y=2-x, \quad x=0$

13.3. $y=\sqrt{4-x^2}, \quad y=0$

13.4. $y=2x+6, \quad x=0, \quad y=0$

13.5. $y=\sqrt{x+4}, \quad y=x+4$

13.6. $x^2+4y^2=16, \quad y=0, \quad x=0$ (1^{st} quadrant)

13.7. $y=\frac{3}{2}x-\frac{1}{2}$, $y=-\frac{2}{3}x+\frac{5}{3}$, $y=\frac{x}{5}+\frac{17}{5}$

13.8. $y=\frac{3}{x}, \quad y=4-x$

13.9. $y=\sin x, \quad y=\cos x, \quad y=0 \ (x\ge 0)$

13.10. $y=\ln x, \quad x=\frac{1}{e}, \quad x=e$

Sketch the following regions.

13.11. $D=\{(x,y): -1\le x\le 0, -x-1\le y\le x+1\}\bigcup$

$$\bigcup\{(x,y): 0\le x\le 1, x-1\le y\le -x+1\}$$

13.12. $D=\left\{(x,y): -1\le x\le 1, \frac{-x-1}{2}\le y\le \frac{x+1}{2}\right\}$

13.13. $D=\left\{(x,y): -2\le x\le 2, x^2-4\le y\le 4-x^2\right\}$

13.14. $D=\left\{(x,y): -1\le x\le 1, x^2-4\le y\le 4-x^2\right\}$

13.15. $D=\left\{(x,y): -3\le x\le 3, -\sqrt{x^2+5}\le y\le \sqrt{x^2+5}\right\}$

13.16. $D=\left\{(x,y): -1\le x\le 0, x\le y\le x^3\right\}\bigcup$

$$\bigcup\left\{(x,y): 0\le x\le 1, x^3\le y\le x\right\}$$

13.2 Double integrals

Definition. A **function of two variables** is a rule by which a value is assigned to z for all (x,y) in a certain region D on the coordinate plane: $z = f(x,y)$. The region D is called the **domain** of f.

Graphically, a function of two variables can be visualised as a *surface* in three-dimensional space.

For positive $f(x,y)$, the definite integral is equal to the *volume* under the surface $z = f(x,y)$ and above the xy-plane for x and y in the region D:

$$V = \iint_D f(M)ds = \iint_D f(x,y)dxdy.$$

By analogy with definite integrals of one variable, the double integral is defined in the following way:

1. Divide the region D into n subregions. Let each subregion have an area of $\triangle S_i$, so that $\sum_{k=1}^{n} \triangle S_i = S$, where S is the area of D. Furthermore, let each subregion have a diameter d_i, where the diameter of a region is understood as the greatest distance between any two points on the boundary of the subregion.
2. Arbitrarily choose a point $M_i = (x_i, y_i)$ from each subregion.
3. Construct the sum $\sum_{i=1}^{n} f(M_i)\triangle S_i = \sum_{i=1}^{n} f(x_i,y_i)\triangle S_i$.
4. Consider the limit as the number of subregions increases without bound, but in such a way that $\max d_i \to 0$. This limit, given that it exists and does not depend on how the subregions were formed or how the points M_i were chosen, is called the **double integral of** $f(x,y)$ **on the region** D:

$$\iint_D f(M)ds = \lim_{\max d_i \to 0} \sum_{i=1}^{n} f(M_i)\triangle S_i.$$

Properties of double integrals

1. $\iint_D kf(x,y)dxdy = k\iint_D f(x,y)dxdy$;
2. $\iint_D \big(f(x,y)+g(x,y)\big)dxdy = \iint_D f(x,y)dxdy + \iint_D g(x,y)dxdy$;
3. If $D = D_1 + D_2$, where D_1 and D_2 do not have any common points, then $\iint_D f(x,y)dxdy = \iint_{D_1} f(x,y)dxdy + \iint_{D_2} f(x,y)dxdy$.

4. If $f(x,y) \leq g(x,y)$ for all $(x,y) \in D$, then $\iint\limits_D f(x,y)dxdy \leq \iint\limits_D g(x,y)dxdy$.

5. If $m \leq f(x,y) \leq M$ for all $(x,y) \in D$ and S is the area of the region D, then $mS \leq \iint\limits_D f(x,y)dxdy \leq MS$.

6. If $f(x,y)$ is continuous on D, then there exists a point $P_0 = (x_0, y_0) \in D$ such that $\iint\limits_D f(x,y)dxdy = f(P_0)S$.

Calculation of double integrals

Let $f(x,y)$ be a function of two variables defined on a region D bounded below and above by $y = \phi_1(x)$ and $y = \phi_2(x)$ and to the left and right by $x = a$ and $x = b$. Then the double integral of $f(x,y)$ over D is equal to

$$\iint\limits_D f(x,y)dxdy = \int\limits_a^b \left(\int\limits_{\phi_1(x)}^{\phi_2(x)} f(x,y)dy \right) dx.$$

The expression in the right side of this equation is called an **iterated integral**. Iterated integrals are also often written in the following forms:

$$\int\limits_a^b \left(\int\limits_{\phi_1(x)}^{\phi_2(x)} f(x,y)dy \right) dx = \int\limits_a^b \int\limits_{\phi_1(x)}^{\phi_2(x)} f(x,y)dydx = \int\limits_a^b dx \int\limits_{\phi_1(x)}^{\phi_2(x)} f(x,y)dy.$$

Note that the limits of integration correspond to the description of the region D in terms of inequalities (those described by vertical lines).

If the region D is bounded to the below and above by $y = c$ and $y = d$ and below and to the left and right by $x = \psi_1(y)$ and $x = \psi_2(y)$, then the double integral of $f(x,y)$ over D is equal to

$$\iint\limits_D f(x,y)dxdy = \int\limits_c^d \left(\int\limits_{\psi_1(y)}^{\psi_2(y)} f(x,y)dx \right) dy.$$

Again, the limits of integration are found by describing the region in terms of inequalities (by horizontal lines). The iterated integral can also be written in the following forms:

$$\int\limits_c^d \left(\int\limits_{\psi_1(y)}^{\psi_2(y)} f(x,y)dx \right) dy = \int\limits_c^d \int\limits_{\psi_1(y)}^{\psi_2(y)} f(x,y)dxdy = \int\limits_c^d dy \int\limits_{\psi_1(y)}^{\psi_2(y)} f(x,y)dx.$$

Note that the order of dx and dy in the iterated integral is of great importance!

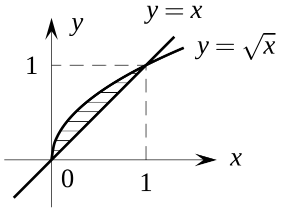
Example 13.4. Change the order of integration in the iterated integral

$$\int_0^1 dx \int_x^{\sqrt{x}} f(x,y)dy.$$

Solution. The region that is being integrated is defined by the inequalities

$$D = \left\{(x,y) : 0 \le x \le 1, x \le y \le \sqrt{x}\right\}.$$

If we sketch the region and consider horizontal lines rather than vertical lines, then we will find:



$$D = \left\{(x,y) : 0 \le y \le 1, y^2 \le x \le y\right\}.$$

Therefore, the iterated integral can be rewritten as

$$\int_0^1 dx \int_x^{\sqrt{x}} f(x,y)dy = \int_0^1 dy \int_{y^2}^{y} f(x,y)dx.$$

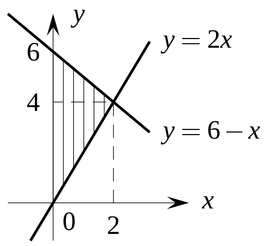
Example 13.5. Change the order of integration in the iterated integral

$$\int_0^4 dy \int_0^{y/2} f(x,y)dx + \int_4^6 dy \int_0^{6-y} f(x,y)dx.$$

Solution. First we will sketch the region D given by

$$D = \left\{(x,y) : 0 \le y \le 4, 0 \le x \le \frac{y}{2}\right\} \bigcup \left\{(x,y) : 4 \le y \le 6, 0 \le x \le 6-y\right\}$$

and consider vertical lines, rather than horizontal lines:



$$D = \{(x,y) : 0 \le x \le 2, 2x \le y \le 6-x\}$$

Therefore, after changing the order of integration we will have

$$\int_0^4 dy \int_0^{y/2} f(x,y)dx + \int_4^6 dy \int_0^{6-y} f(x,y)dx = \int_0^2 dx \int_{2x}^{6-x} f(x,y)dy.$$

Change the order of integration of the following iterated integrals.

13.17. $\int_0^1 dx \int_0^x f(x,y)dy$

13.18. $\int_0^4 dy \int_{y/2}^2 f(x,y)dx$

13.19. $\int_1^2 dx \int_1^{2x-1} f(x,y)dy$

13.20. $\int_1^2 dx \int_0^{\ln x} f(x,y)dy$

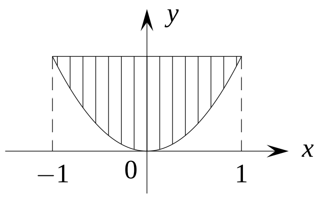
13.21. $\int_1^6 dx \int_{30/x}^{31-x} f(x,y)dy$

13.22. $\int_0^1 dy \int_{1-y}^{1+y} f(x,y)dx$

13.23. $\int_{-1}^1 dx \int_{-x}^{1+x} f(x,y)dy$

Example 13.6. Find the double integral of $f(x,y) = x+y$ over D, where D is the region bounded by the graphs of $y = x^2$ and $y = 1$.

Solution. First we need to rewrite the double integral as an iterated integral. The boundaries of the iterated integral can be found by describing D in terms of inequalities. For instance, if we consider vertical lines, then we will have:



$$D = \left\{(x,y) : -1 \le x \le 1, x^2 < y < 1\right\}.$$

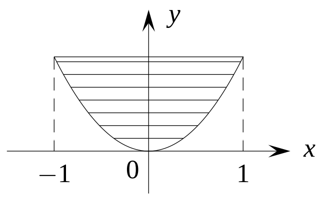
The integral will then equal:

$$\iint_D f(x,y)dxdy = \int_{-1}^1 dx \int_{x^2}^1 (x+y)dy = \int_{-1}^1 \left(\int_{x^2}^1 (x+y)dy\right) dx =$$

$$= \int_{-1}^1 \left(xy + \frac{y^2}{2}\right)\bigg|_{x^2}^1 dx = \int_{-1}^1 \left(x + \frac{1}{2} - x^3 - \frac{x^4}{2}\right) dx =$$

$$= \left(\frac{x^2}{2} + \frac{x}{2} - \frac{x^4}{4} - \frac{x^6}{10}\right)\bigg|_{-1}^1 = \frac{4}{5}.$$

It is possible of course to describe D in a different way:



$$D = \left\{(x,y) : 0 \le y \le 1, -\sqrt{y} < x < \sqrt{y}\right\}.$$

The integral will equal:

$$\iint_D f(x,y)dxdy = \int_0^1 dy \int_{-\sqrt{y}}^{\sqrt{y}} (x+y)dx = \int_0^1 \left(\int_{-\sqrt{y}}^{\sqrt{y}} (x+y)dx\right) dy =$$

$$= \int_0^1 \left(\frac{x^2}{2} + xy \right) \Big|_{-\sqrt{y}}^{\sqrt{y}} dy = \int_0^1 \left(\frac{y}{2} + y^{3/2} - \frac{y}{2} + y^{3/2} \right) dy = 2\,\frac{2}{5} y^{5/2} \Big|_0^1 = \frac{4}{5}.$$

The two results are, of course, exactly the same.

Find the following double integrals:

13.24. $\iint\limits_D xydxdy, \quad D = \left\{(x,y) : -1 \le x \le 2, -x^2 \le y \le x+1\right\}$

13.25. $\iint\limits_D (x^2+y^2)dxdy, \quad D = \left\{(x,y) : -1 \le x \le 1, 2x^2 \le y \le x^2+1\right\}$

13.26. $\iint\limits_D \ln y dxdy, \quad D = \left\{(x,y) : e \le y \le e^2, y \le x \le 2y\right\}$

13.27. $\iint\limits_D e^{x/y} dxdy, \quad D = \left\{(x,y) : 1 \le y \le 2, y \le x \le y^3\right\}$

13.28. $\iint\limits_D (4xy - y^3)dxdy,$ D is the region bounded by $y = \sqrt{x}$ and $y = x^3$

13.29. $\iint\limits_D (6x^2 - 40y)dxdy,$ D is the triangle with vertices $(0,3)$, $(1,1)$, $(5,3)$

Find the following iterated integrals after reversing the order of integration.

13.30. $\int_0^3 dx \int_{x^2}^9 x^3 e^{y^3} dy$

13.31. $\int_0^8 dy \int_{\sqrt[3]{y}}^2 \sqrt{x^4+1}dx$

ANSWERS

Chapter 1.

1.1. $x \neq -4$. **1.2.** $0 \leq x \leq 4$. **1.3.** $0 < x < 4$. **1.4.** $x > 2$. **1.5.** $-2 \leq x < 1$, $x \neq 0$. **1.6.** $-4 \leq x \leq 4$. **1.7.** $2n\pi \leq x \leq (2n+1)\pi$, $n \in \mathbb{Z}$. **1.8.** $-\frac{1}{3} \leq x \leq 1$. **1.9.** $x > 1$ and $\frac{1}{2n+1} < x < \frac{1}{2n}$, $n = \pm 1, \pm 2, \ldots$. **1.10.** $x > 1$. **1.11.** No. **1.12.** No. **1.13.** No. **1.14.** No. **1.15.** No. **1.16.** No. **1.17.** $-1 \leq x \leq 2$; $0 \leq y \leq 1.5$. **1.18.** $2n\pi + \frac{\pi}{3} < x < 2n\pi + \frac{5\pi}{3}$, $n \in \mathbb{Z}$; $-\infty < y \leq \log 3$. **1.19.** $-\infty < x < \infty$; $0 \leq y \leq \pi$. **1.20.** $1 \leq x \leq 100$; $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$. **1.58.** They are equivalent. **1.59.** They are equivalent. **1.60.** They are equivalent. **1.63.** Yes; $y = 0$. **1.64.** Not necessarily.

Chapter 2.

2.10. $1/2$. **2.11.** ∞. **2.12.** $1/24$. **2.13.** 0. **2.14.** $7/9$. **2.15.** $4/5$. **2.16.** $64/27$. **2.17.** $\dfrac{1}{1+\sqrt{2}}$. **2.18.** 0. **2.19.** $\ln 3/\ln 2$. **2.20.** $1/2$. **2.21.** 3. **2.22.** $\frac{1-b}{1-a}$. **2.23.** $-1/2$. **2.24.** 1. **2.25.** $\sqrt{2}$. **2.26.** 1. **2.27.** 0. **2.28.** 4. **2.29.** 0. **2.30.** $1/4$. **2.31.** $1/2$. **2.32.** $1/2$. **2.33.** $\sqrt{3}$. **2.34.** 1. **2.35.** e^2. **2.36.** e^3. **2.37.** e^4. **2.38.** $e^{-1/3}$. **2.39.** $x_n + y_n$ does not converge; $x_n y_n$ may or may not converge. **2.40.** $x_n + y_n$ and $x_n y_n$ may or may not converge. **2.41.** No. **2.42.** No; No. **2.43.** No. **2.47.** 1.

Chapter 3.

3.1. 1. **3.2.** $3/4$. **3.3.** 1. **3.4.** 0. **3.5.** -1/4. **3.6.** -4. **3.7.** 1. **3.8.** $2/3$. **3.9.** $1/2$. **3.10.** $-2/5$. **3.11.** 0. **3.12.** $-9/4$. **3.13.** $1/2$. **3.14.** 1. **3.15.** 29. **3.16.** $1/8$. **3.17.** $\left(\frac{2}{3}\right)^5$. **3.18.** 32. **3.19.** 6. **3.20.** $1/3$. **3.21.** 2. **3.22.** $-1/3$. **3.23.** $1/2$. **3.24.** 3. **3.25.** $1/4$. **3.26.** $1/3$. **3.27.** -2. **3.28.** $1/40$. **3.29.** $1/6$. **3.30.** $-1/6$. **3.31.** $-1/2$. **3.32.** $3/2$. **3.33.** $1/2$. **3.34.** $112/27$. **3.35.** -45. **3.36.** $3/2$. **3.37.** n/m. **3.38.** 0. **3.39.** $1/2$. **3.40.** $3/2$. **3.41.** 2. **3.42.** 0.5. **3.43.** $1/3$. **3.44.** $3/2$. **3.45.** e^2. **3.46.** e. **3.47.** e^4. **3.48.** e. **3.49.** $1/e$. **3.50.** e^2. **3.51.** e^3. **3.52.** 0. **3.53.** e^3. **3.54.** $\sqrt{3}/3$. **3.55.** 0. **3.56.** 0. **3.57.** $\lim\limits_{x\to\infty}(f(x)+g(x))$ does not exist; $\lim\limits_{x\to\infty}(f(x)g(x))$ may or may not exist. **3.58.** No. **3.59.** No. **3.60.** -1; 1. **3.61.** $-\infty$; ∞. **3.62.** ∞; ∞. **3.63.** -5; 5. **3.64.** 1; -2. **3.65.** $y = x$, $y = -x$. **3.66.** $x = 0$, $y = 1$. **3.67.** $y = 0$. **3.68.** $x = 0$, $y = x + 6$. **3.69.** $x = -3$, $y = x - 3$. **3.70.** $y = x - 1$. **3.71.** $x = -3$, $x = 1$, $y = x - 2$. **3.72.** $y = x$. **3.73.** $y = -x$, $y = x$. **3.74.** $y = \pm 1$. **3.75.** $x = \pm 1$, $y = \pm x$. **3.76.** $y = x$. **3.77.** $x = 0$. **3.78.** $x = 0$, $y = 1$. **3.79.** $x = 2$, $y = x + 1$, $y = -x - 1$. **3.80.** $y = x - 2$. **3.81.** 3. **3.82.** $3/2$. **3.83.** $-2/5$. **3.84.** $3/7$. **3.85.** $1/2$. **3.86.** 1. **3.87.** $4/9$. **3.88.** $2/3$. **3.89.** $1/3$. **3.90.** -2. **3.91.** $-3/2$. **3.92.** $1/(2\pi)$. **3.93.** $1/2$. **3.94.** $-\sqrt{2}/4$. **3.95.** $-1/2$. **3.96.** $-1/\pi$. **3.97.** 2. **3.98.** $\sqrt{2}/2$. **3.99.** $-1/2$. **3.100.** -2. **3.101.** 1. **3.102.** 1. **3.103.** $2/\ln a$. **3.104.** $\ln a/3$. **3.105.** $-4/3$. **3.106.** $1/2$. **3.107.** $1/4$. **3.108.** $\tan 4$. **3.109.** $1/3$. **3.110.** 1. **3.111.** $2\ln 3$. **3.112.** 1. **3.113.** $-\sin x$. **3.114.** $-\cos x$.

Chapter 4.

4.1. 0.5. **4.2.** 1.5. **4.3.** 1. **4.4.** 2. **4.5.** 2. **4.6.** 0. **4.7.** e. **4.8.** 0. **4.9.** 3. **4.10.** It is not possible to define $f(x)$ so that $f(x)$ is continuous at $x=1$. **4.11.** $m=1$. **4.12.** $k=1.2$. **4.13.** $A=6$. **4.14.** $a=-1$. **4.15.** $b=\frac{5}{\ln 2}$. **4.16.** $d=\ln 2$. **4.17.** $k=2$. **4.18.** $a=-4$. **4.19.** $c=-6$ or $c=1$. **4.20.** $n=\ln 2$. **4.21.** $a=3$, $b=1$. **4.22.** $a=-1$, $b=1$. **4.23.** No. **4.25.** $f(x)+g(x)$ is not continuous at $x=x_0$; $f(x)g(x)$ may or may not be continuous at $x=x_0$. **4.26.** $f(x)+g(x)$ and $f(x)g(x)$ may or may not be continuous at $x=x_0$. **4.30.** 1 $(x=0)$. **4.31.** Type I at $x=0$. **4.32.** Type II at $x=-4$. **4.33.** Type II at $x=-4$. **4.34.** Type I at $x=2$. **4.35.** Type I at $x=1$. **4.36.** Removable at $x=2$, type II at $x=3$. **4.37.** Type II at $x=1$ and $x=-2$. **4.38.** Type I at $x=2$, type II at $x=-2$. **4.39.** Type I at $x=0$, type II at $x=1$. **4.40.** Type II at $x=1$. **4.41.** Type I at $x=0$. **4.42.** Removable at $x=0$. **4.43.** Removable at $x=0$. **4.44.** Type II at $x=0$. **4.45.** Type I at $x=-1$. **4.46.** Type I at $x=0$. **4.47.** Type I at $x=0$. **4.48.** Removable at $x=0$, type II at $x=\pm 1$. **4.49.** Type II at $x=k$, $k\in\mathbb{Z}$. **4.50.** Type I at $x=\frac{(2k+1)\pi}{2}$, $k\in\mathbb{Z}$. **4.51.** Jump at $x=0,\ 1,\ 2$. **4.55.** No.

Chapter 5.

5.1. 1. **5.2.** $-\frac{1}{x^2}$. **5.3.** $3x^2$. **5.4.** $2x+2$. **5.5.** $\frac{1}{2\sqrt{x}}$. **5.6.** $2\cos\frac{x}{2}$. **5.7.** $-\frac{3}{2\sqrt{x^3}}+2\cdot 3^x\ln 3$. **5.8.** $\frac{1}{x\ln a}$. **5.9.** $\frac{1}{x\ln 2}$. **5.10.** $\frac{3}{2\sqrt{x}}+\frac{1}{x\ln 2}$. **5.11.** $f'(3)\approx 1$, 1.1 or 1.2; $f''(3)\approx 4$. **5.12.** $f'(-1.5)=1$, $f'(1)=0$, $f'(2.5)=-4$. **5.13.** $10x+4$. **5.14.** $10(2x+4)(x^2+4x-12)^9$. **5.15.** $\frac{1}{(3+x)^2}$. **5.16.** $-\frac{8x}{(x^2-4)^5}$. **5.17.** $-\frac{2x^2+2x-1}{(2x^2+4x+3)^2}$. **5.18.** $-2\frac{x^2-x-1}{(1+x^2)^2}$. **5.19.** $\frac{6x^2(x^4+1)^2(x^4+2x-1)}{(x^3+1)^3}$. **5.20.** $2\frac{x^4-2x^3-2x^2-2x+1}{(x^3+2x^2+1)^2}$. **5.21.** $-\frac{x}{\sqrt{4-x^2}}$. **5.22.** $\frac{3x^3}{\sqrt{x^2-3}}$. **5.23.** $\frac{2x^2}{\sqrt[3]{x^3+1}}$. **5.24.** $\frac{9x-18}{2\sqrt{2-3x}(3x-2)}$. **5.25.** $\frac{1}{\sqrt[3]{(x^3+1)^4}}$. **5.26.** $\frac{4x^2-4x-2}{3(x-1)\sqrt[3]{x^2-x}}$. **5.27.** $-\frac{x-3\sqrt[3]{x^2}+4\sqrt[4]{x^3}}{12\sqrt[12]{x^{17}}(\sqrt[3]{x}+1)^2}$. **5.28.** $-\frac{1}{(x-1)^2}\sqrt{\frac{x-1}{x+1}}$. **5.29.** $-\frac{1}{6\sqrt[3]{x^2}\sqrt{(\sqrt[3]{x}+1)^3}}$. **5.30.** $-\frac{1}{2\sqrt{x^2-1}(\sqrt{x+1}+\sqrt{x-1})}$. **5.31.** $x^5(2-5x^3)^{2/3}$. **5.32.** $\frac{4\sin x}{(\cos x+2)^2}$. **5.33.** $\frac{1}{\sin^2 x}$. **5.34.** $-(8x+3)\sin(4x^2+3x+4)$. **5.35.** $-\frac{\sin x}{2\sqrt{\cos x}}$. **5.36.** $\frac{\sin x}{2\sqrt{\cos^3 x}}$. **5.37.** $\frac{\sin 2x}{(\cos^2 x-2)^2}$. **5.38.** $\frac{3x^2}{\cos^2(x^3)}$. **5.39.** $-\frac{1+\cot^2 x}{2\sqrt{\cot x}}$. **5.40.** $(2x+1+\tan^2 x)\cos(x^2+\tan x)$. **5.41.** $\frac{\cos x}{\cos^2(\sin x)}$. **5.42.** $\frac{\sqrt{\tan^3 x}}{\cos^4 x}$. **5.43.** $\tan^4 x$. **5.44.** $\tan^2\frac{x}{2}$. **5.45.** $-\frac{1}{\sqrt{1-x^2}}$. **5.46.** $\frac{x^3}{1+x^4+x^8}$. **5.47.** $\frac{\cos x}{1+\sin^2 x}$. **5.48.** $\frac{1}{2\sqrt{3x-x^2-2}}$. **5.49.** $\frac{1}{(x+1)\sqrt{x}}$. **5.50.** $-\frac{2x}{x^4+1}$. **5.51.** 0. **5.52.** $\frac{2x}{x^4+1}$. **5.53.** $\frac{2x^2}{\sqrt{1-x^2}}$. **5.54.** $\frac{2x^2}{(1+x^2)^2}$. **5.55.** $\frac{\sin x\cos x}{1+\sin^4 x}$. **5.56.** $x(1+x^2)\cot^{-1}x$. **5.57.** $4\sqrt{1-x^2}\sin^{-1}x$. **5.58.** $\frac{4x+1}{2x^2+x+1}$. **5.59.** $-\frac{4x}{x^4-1}$. **5.60.** $\frac{1}{2(x+1)}$. **5.61.** $\frac{6x}{1+x^2}\ln^2(1+x^2)$. **5.62.** $\cot x$. **5.63.** $-\frac{2}{x\ln^2(2x)}$. **5.64.** $-\frac{1}{\sqrt{1+x^2}}$. **5.65.** $-\frac{1}{x^2-1}$. **5.66.** $\frac{1}{x\sqrt{1+x^2}}$. **5.67.** $\frac{1}{\sqrt{x}(x-1)}$. **5.68.** $\frac{x+1}{x(x+\ln x)}$. **5.69.** $x\ln(4+x^4)$. **5.70.** $e^{2x}(2x+1)$. **5.71.** $\frac{3^x\ln 3}{(3^x+1)^2}$. **5.72.** $\frac{xe^x}{(x+1)^2}$. **5.73.** $\frac{\ln(x+1)-\ln x}{x(1+x)}$. **5.74.** $x\ln\frac{1+x}{1-x}$. **5.75.** $\frac{\sin x+\sin^3 x}{\cos 2x}$. **5.76.** $\sin x\ln\tan x$. **5.77.** $2e^x\cos x$. **5.78.** $(x+1)^{x^2+1}\left(2x\ln(x+1)+\frac{x^2+1}{x+1}\right)$. **5.79.** $(\sin x)^x(\ln\sin x+x\cot x)$. **5.80.** $(\sin x)^{\sin x}\cos x(\ln\sin x+1)$. **5.81.** $(\tan x)^{\ln x}\left(\frac{1}{x}\ln\tan x+\ln x\frac{2}{\sin 2x}\right)$. **5.82.** $3x^2\,\mathrm{sign}(x)$. **5.83.** $3\sin^2 x\cos x\,\mathrm{sign}(\sin x)$. **5.84.** $f(x)+g(x)$ is not differentiable at $x=x_0$; $f(x)g(x)$ may or may not be differentiable at $x=x_0$. **5.85.** $f(x)+g(x)$ and $f(x)g(x)$ may or may not be differentiable at $x=x_0$. **5.93.** $a=4$, $b=-4$. **5.94.** $a=2x_0$, $b=-x_0^2$. **5.95.** $a=f'(x_0)$, $b=f(x_0)-x_0f'(x_0)$. **5.96.** $\frac{1-(n+1)x^n+nx^{n+1}}{(1-x)^2}$. **5.97.** $\frac{2-n(n+1)x^{n-1}+2(n^2-1)x^n-n(n-1)x^{n+1}}{(1-x)^3}$. **5.98.** $\tan^{-1}\frac{1+x}{1-x}=\tan^{-1}x+\frac{\pi}{4}$ for $x<1$ and $\tan^{-1}\frac{1+x}{1-x}=\tan^{-1}x-\frac{3\pi}{4}$ for $x>1$. **5.100.** $\frac{1}{6}$. **5.101.** 1. **5.102.** $\frac{1}{2}$. **5.103.** $-2\ln 2$. **5.104.** $\frac{1-x-y}{x-y}$. **5.105.** $-\frac{b^2x}{a^2y}$. **5.106.** $\frac{ay-x^2}{y^2-ax}$. **5.107.** $\frac{2a}{3(1-y^2)}$. **5.108.** $-\frac{1+y\sin(xy)}{x\sin(xy)}$. **5.109.** $-\sqrt[3]{\frac{y}{x}}$. **5.110.** $\frac{1+y^2}{y^2}$. **5.111.** $\frac{x+y}{x-y}$. **5.112.** $\frac{1}{2}$. **5.115.** $(0,0)$; $(1,1)$; $(2,0)$. **5.116.** $y=2x+2$; $y=2x-2$. **5.117.** $y=\frac{\sqrt{2}}{2}\left(\frac{3\pi}{4}+1-x\right)$. **5.118.** $y=\cos x_0(x-x_0)+y_0$. **5.119.** $x+2y=4a$. **5.120.** $(1,3)$; $(3,27)$. **5.121.** $2x-y+1=0$. **5.123.** $x+25y=0$, $y=-x$. **5.124.** $\frac{1}{2e}$. **5.125.** $\frac{\pi}{4}$. **5.126.** $y=mx-am^2$. **5.127.** $27x-3y-79=0$. **5.128.** $(1,0)$; $(-1,-4)$. **5.129.** $x-y-3e^{-2}=0$. **5.130.** $3x+y+6=0$. **5.131.** a) $y=\sqrt[3]{4}(x+1)$, $y=-\frac{1}{\sqrt[3]{4}}(x+1)$; b) $y=3$, $x=2$; c) $x=3$, $y=0$. **5.132.** $y=\frac{x-5}{4}$, $y=3-4x$. **5.134.** 90°. **5.135.** $\frac{241}{22}$. **5.136.** $\frac{323}{108}$. **5.137.** $1+\frac{1}{10\ln 10}$. **5.138.** $\frac{\sqrt{3}\pi}{200}+\frac{1}{2}$. **5.139.** $-\left(\frac{\sqrt{3}}{2}+\frac{\pi}{360}\right)$. **5.140.** $\frac{1}{3}+\frac{2}{125}$. **5.143.** Three roots, located on $(1,2)$; $(2,3)$; $(3,4)$. **5.146.** $a(1-\ln a)-b(1-\ln b)=(b-a)\ln c$, $a<c<b$. **5.147.** 1. **5.148.** $2\pm\frac{2\sqrt{3}}{3}$. **5.149.** $\frac{3\pm\sqrt{5}}{2}$. **5.150.** 0.5; $\sqrt{2}$.

Chapter 6.

6.1. 0.6. **6.2.** $\frac{a}{b}$. **6.3.** 2. **6.4.** $\frac{1}{3}$. **6.5.** $a^a(\ln a-1)$. **6.6.** 1. **6.7.** $\frac{3}{e}$. **6.8.** 0. **6.9.** 2. **6.10.** $\frac{1}{a}$. **6.11.** 0. **6.12.** 0. **6.13.** $\frac{1}{6}$. **6.14.** 0.5. **6.15.** 0.5. **6.16.** $\frac{9}{50}$. **6.17.** $\frac{\ln a}{6}$. **6.18.** $\frac{2}{3}$. **6.19.** 18. **6.20.** $\frac{1}{2}$. **6.21.** 2. **6.22.** $e^{1/3}$. **6.23.** 1. **6.24.** $\frac{1}{6}$. **6.25.** $\frac{2}{3}$. **6.26.** $e^{1/6}$. **6.27.** $\frac{1}{\sqrt{e}}$. **6.28.** $\frac{1}{e}$. **6.31.** $f''(x)$. **6.32.** Increasing on $(-1,1)$; decreasing on $(-\infty,-1)$ and $(1,\infty)$. **6.33.** Increasing on $(-1,0)$ and $(1,\infty)$; decreasing on $(-\infty,-1)$ and $(0,1)$. **6.34.** Increasing on $(-\infty,-0.5)$ and $(\frac{11}{18},\infty)$; decreasing on $(-0.5,\frac{11}{18})$. **6.35.** Increasing on $(-1,1)$; decreasing on $(-\infty,-1)$ and $(1,\infty)$. **6.36.** Increasing on $(0,100)$; decreasing on $(100,\infty)$. **6.37.** Increasing on $(-\infty,0)$; decreasing on $(0,\infty)$. **6.38.** Increasing on $(-\infty,0)$; decreasing on $(0,\infty)$. **6.39.** Increasing on $(0,2)$; decreasing on $(-\infty,0)$ and $(2,\infty)$. **6.40.** Increasing on $(-1,0)$ and $(1,\infty)$; decreasing on $(-\infty,-1)$ and $(0,1)$. **6.41.** Increasing on (e,∞); decreasing on $(0,1)$ and $(1,e)$. **6.42.** Increasing on $(-\infty,\infty)$. **6.43.** Increasing on $(0,\frac{2}{\ln 2})$; decreasing on $(-\infty,0)$ and $(\frac{2}{\ln 2},\infty)$. **6.44.** Increasing on $(0,\frac{\pi}{6})$, $(\frac{\pi}{2},\frac{5\pi}{6})$ and $(\frac{3\pi}{2},2\pi)$; decreasing on $(\frac{\pi}{6},\frac{\pi}{2})$ and $(\frac{5\pi}{6},\frac{3\pi}{2})$. **6.45.** Increasing on $(\frac{k\pi}{2},\frac{k\pi}{2}+\frac{\pi}{3})$; decreasing on $(\frac{k\pi}{2}+\frac{\pi}{3},\frac{k\pi}{2}+\frac{\pi}{2})$, $k\in\mathbb{Z}$. **6.46.** $-0.1C$. **6.47.** 7m.p.h. **6.48.** $5\sqrt{41}$ m.p.h. **6.49.** $-\frac{2\sqrt{3}}{3}\frac{\text{ft}}{\text{sec}}$. **6.50.** $\frac{1}{80\pi}\frac{\text{ft}}{\text{sec}}$. **6.51.** $-648\frac{\text{in}^3}{\text{sec}}$. **6.52.** $\frac{4}{9}\frac{\text{cm}}{\text{sec}}$. **6.53.** $25\frac{\text{ft}^2}{\text{sec}}$ and $0.4\frac{\text{ft}}{\text{sec}}$. **6.54.** $-0.6\frac{\text{rad}}{\text{sec}}$. **6.55.** $\frac{3}{\pi}\frac{\text{ft}}{\text{sec}}$. **6.56.** $0.116\frac{\text{ohm}}{\text{sec}}$. **6.57.** 10π mm^2/sec. **6.58.** Concave down on $(-\infty,\frac{5}{3})$; concave up on $(\frac{5}{3},\infty)$; inflection point at $x=\frac{5}{3}$. **6.59.** Concave up on $(-\infty,\infty)$; no inflection points. **6.60.** Concave down on $(2,4)$; concave up on $(-\infty,2)$ and $(4,\infty)$; inflection points at $x=2$ and $x=4$. **6.61.** Concave down on $(-\infty,0)$; concave up on $(0,\infty)$; inflection point at $x=0$. **6.62.** Concave up on $(-\infty,\infty)$; no inflection points. **6.63.** Concave up on $(-\infty,\infty)$; no inflection points.

6.64. Concave down on $(-3a,0)$ and $(3a,\infty)$; concave up on $(-\infty,-3a)$ and $(0,3a)$; inflection points at $x=\pm 3a$ and $x=0$. **6.65.** Concave down on $(-\infty,-1)$ and $(1,\infty)$; concave up on $(-1,1)$; inflection points at $x=\pm 1$. **6.66.** Concave down on $(-\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2})$; concave up on $(-\infty,-\frac{\sqrt{2}}{2})$ and $(\frac{\sqrt{2}}{2},\infty)$; inflection points at $x=\pm\frac{\sqrt{2}}{2}$. **6.67.** Concave up on $(0,\infty)$; no inflection points. **6.68.** Concave down on $(2\pi k, 2\pi k+\pi)$, $k\in\mathbb{Z}$. **6.72.** $\frac{1}{2\sigma}$. **6.73.** $a=-1.5$, $b=4.5$. **6.74.** $a\le -\frac{e}{6}$ and $a>0$. **6.77.** Minimum $y=0$ at $x=0$; maximum $y=1$ at $x=\pm 1$. **6.78.** Minimum $y=-1$ at $x=-1$; maximum $y=1$ at $x=1$. **6.79.** Minimum $y=0$ at $x=2$; maximum $y=3$ at $x=1$. **6.80.** Minimum $y=-\frac{1}{24}$ at $x=\frac{7}{5}$. **6.81.** Minimum $y=\frac{8}{3}$ at $x=-2$; maximum $y=4$ at $x=0$. **6.82.** Minimum $y=-\frac{3\sqrt[3]{2}}{8}$ at $x=\frac{3}{4}$. **6.83.** Local maximum $y=1$ at $x=1$. **6.84.** Maximum $y=\frac{1}{\ln 3}$ at $x=-3$. **6.85.** Maximum $y=\frac{\pi}{4}-\frac{\ln 2}{2}$ at $x=1$. **6.86.** 2; 66. **6.87.** 0.75; 13. **6.88.** 2; 6. **6.89.** -10; 2. **6.90.** -1; 0.6. **6.91.** 0.6; 1. **6.92.** 1; 3. **6.93.** $-\frac{\pi}{2}$; $\frac{\pi}{2}$. **6.94.** $\ln 3$; $\ln 19$. **6.95.** 0; 132. **6.96.** 0; $\sqrt[3]{9}$. **6.97.** 0; $\frac{\pi}{4}$. **6.98.** $\left(\frac{1}{e}\right)^{1/e}$; no maximum value. **6.99.** 3 and 6. **6.100.** $21\frac{\text{m}}{\text{sec}^2}$. **6.101.** $y=\frac{9-3x}{4}$; $y=\frac{3x+9}{4}$. **6.102.** 5ft and 5ft. **6.103.** a) $\frac{4\cdot 20^3\pi}{27}$; b) 800π. **6.104.** 30° and 60°. **6.105.** $\frac{2\sqrt{3}\pi L^3}{27}$. **6.106.** 20. **6.107.** a) 30 m.p.h.; b) 30.373 m.p.h.. **6.108.** 2m sides and 6m height. **6.109.** 15. **6.110.** 500. **6.111.** $\sqrt[3]{2}$ and 6. **6.112.** $\sqrt{3}$. **6.113.** $2ab$. **6.114.** 4cm and 6cm. **6.115.** $\sqrt{\frac{aS}{b}}$ and $\sqrt{\frac{bS}{a}}$. **6.116.** $\sqrt{300}$ feet. **6.117.** $\frac{28\sqrt{13}}{13}\frac{\text{km}}{\text{min}}$. **6.118.** $\frac{\sqrt{2}}{2}$. **6.123.** a) 1; b) 1; c) 5. **6.124.** a) The solid line is the graph of $h'(x)$. **6.127.** The function is odd. Maximum $y(1)=0.5$, minimum $y(-1)=-0.5$. Inflection points at $x=-\sqrt{3}$, $x=0$, $x=\sqrt{3}$. Asymptote $y=0$. **6.128.** Not defined at $x=\pm 1$. The function is odd. No extremes. Inflection point at $x=0$. Asymptotes $x=-1$, $x=1$, $y=0$. **6.129.** Not defined at $x=0$. Minimum $y(0)=0$, no maximums. Inflection point at $x=-\sqrt[3]{2}$. Asymptote $x=0$. **6.130.** Not defined at $x=-1$. Maximum $y(-1)=\frac{2}{27}$, minimum $y(1)=0$. Inflection points at $x=5\pm 2\sqrt{3}$. Asymptotes $x=-1$ and $y=0$. **6.131.** Not defined at $x=0$. Maximums $y(1)=\frac{7}{2}$ and $y(-3)=-\frac{11}{6}$, minimum $y(2)=\frac{27}{8}$. Inflection point at $x=\frac{9}{7}$. Asymptotes $x=0$ and $y=0.5x+1$. **6.132.** Maximum $y(1)=\frac{1}{e}$, no minimums. Inflection point at $x=2$. Asymptote $y=0$. **6.133.** The function is even. Minimum $y(0)=0$, no maximums. Inflection points at $x=\pm 1$. No asymptotes. **6.134.** Defined for $x>0$. No extremes. Inflection point at $x=e^{3/2}$. Asymptotes $x=0$ and $y=x$. **6.135.** Symmetrical with respect to the line $x=1$. Maximum $y(1)=e$, no minimums. Inflection points at $x=1\pm\frac{\sqrt{2}}{2}$. Asymptote $y=0$. **6.136.** Not defined at $x=0$. Maximum $y(-1)=\frac{1}{e}$, minimum $y(2)=4\sqrt{e}$. Inflection point at $x=-\frac{2}{5}$. Asymptotes $x=0$ and $y=x+3$. **6.137.** The function is odd. Maximum $y(-1)=\frac{\pi}{2}-1$, minimum $y(1)=1-\frac{\pi}{2}$. Inflection point at $x=0$. Asymptotes $y=x\pm\pi$. **6.138.** Maximum $y(\frac{7}{11})=2.2$, minimum $y(1)=0$. Inflection points at $x=-1$ and $x=\frac{7\pm 3\sqrt{3}}{11}$. No asymptotes. **6.139.** Defined for $-1\le x\le 1$. The graph is symmetrical with respect to both axes. Maximum and minimum $|y|=0.5$ at $x=\pm\frac{\sqrt{2}}{2}$. Inflection point at $x=0$. No asymptotes. **6.140.** Defined for $-1\le x\le 1$. The graph is symmetrical with respect to both axes. Maximum and minimum $|y|=1$ at $x=0$. Inflection points at $\pm\frac{\sqrt{2}}{2}$. No asymptotes.. **6.141.** Not defined at $x=\frac{\pi}{2}+k\pi$, $k\in\mathbb{Z}$. Periodic with period π. No extremes.

No inflection points. Asymptotes $x = \frac{\pi}{2} + k\pi$. **6.142.** The function is even. Maximum $y(0) = 3$, minimums $y(\pm 2) = -1$. No inflection points. No asymptotes. **6.143.** The function is even. Minimum $y = 0$ at $x = 0$, no maxima. No inflection points. Asymptote $y = 1$. **6.144.** The function is odd. No extremes. Inflection point at $x = 0$. No asymptotes. **6.145.** Function defined for $x \le 0$ and $x \ge \frac{2}{3}$. Maximum $y(\frac{2}{3}) = \pi$, minimum $y(0) = 0$. No inflection points. Asymptote $y = \frac{\pi}{3}$.

Chapter 7.

7.7. Divergent. **7.8.** Divergent. **7.9.** Divergent. **7.10.** Divergent. **7.11.** Divergent. **7.12.** Divergent. **7.13.** Convergent. **7.14.** Divergent. **7.15.** Convergent. **7.16.** Divergent. **7.17.** Convergent. **7.18.** Convergent. **7.19.** Divergent. **7.20.** Convergent. **7.21.** Divergent. **7.22.** Convergent. **7.23.** Divergent. **7.24.** Convergent. **7.25.** Divergent. **7.26.** Convergent. **7.27.** Convergent. **7.28.** Convergent. **7.29.** Convergent. **7.30.** Convergent. **7.31.** Divergent. **7.32.** Divergent. **7.33.** Convergent. **7.34.** Divergent. **7.35.** Convergent. **7.36.** Convergent. **7.37.** Divergent. **7.38.** Convergent. **7.39.** Convergent. **7.40.** All of these series may be either convergent or divergent. **7.41.** No. **7.43.** Converges absolutely. **7.44.** Converges absolutely. **7.45.** Converges absolutely. **7.46.** Divergent. **7.47.** Converges conditionally. **7.48.** Converges absolutely. **7.49.** Converges absolutely. **7.50.** Converges conditionally. **7.51.** Converges absolutely. **7.52.** Converges absolutely. **7.53.** Converges absolutely. **7.54.** Converges absolutely. **7.55.** Converges conditionally. **7.56.** Расходится. **7.57.** Converges absolutely. **7.58.** Converges absolutely. **7.59.** Converges conditionally. **7.60.** Converges absolutely. **7.61.** Converges conditionally. **7.63.** $(-1,1)$. **7.64.** $(-1/3,1/3)$. **7.65.** $[-9,-7]$. **7.66.** $[-2,2)$. **7.67.** $[-1,1]$. **7.68.** $[-1,1]$. **7.69.** $[-6,-2)$. **7.70.** $[2,4]$. **7.71.** $[-5,3)$. **7.72.** $[-1,1)$. **7.73.** $[1,3]$. **7.74.** $[-3,-1]$. **7.75.** $(4,6]$. **7.76.** $(-4,4)$. **7.77.** $(-3,3)$. **7.78.** $(-6,-4)$. **7.79.** $(-9,9]$. **7.80.** $[-.5,.5]$. **7.81.** $(-\infty,\infty)$. **7.82.** $(-1,1)$.

Chapter 8.

8.1. $3(x-1)^3 + 11(x-1)^2 + 14(x-1) + 6$. **8.2.** $2 - 2(x-1) + 2(x-1)^2 - 2(x-1)^3$.
8.3. $1 - x^2 + \frac{1}{2!}x^4 - \frac{1}{3!}x^6 + \ldots + \frac{(-1)^n}{n!}x^{2n} + \ldots$.
8.4. $\frac{1}{2}x - \frac{1}{2^3 \cdot 3!}x^3 + \ldots + (-1)^{n+1}\frac{1}{2^{2n-1}(2n-1)!}x^{2n-1} + \ldots$.
8.5. $1 - x^2 + \frac{2^3}{4!}x^4 + \ldots + (-1)^n \frac{2^{2n-1}}{(2n)!}x^{2n} + \ldots$.
8.6. $-\frac{2}{3!}x^3 + \frac{4}{5!}x^5 - \ldots + (-1)^n \frac{2n}{(2n+1)!}x^{2n+1} + \ldots$.
8.7. $\ln 10 + \frac{1}{10}(x-10) - \frac{1}{2 \cdot 10^2}(x-10)^2 + \ldots + (-1)^{n+1}\frac{1}{n \cdot 10^n}(x-10)^n + \ldots$.
8.8. $(x-1)^2 - \frac{1}{2}(x-1)^3 + \ldots + (-1)^{n+1}\frac{1}{n}(x-1)^{n+1} + \ldots$.
8.9. $1 + \frac{1}{2}x^2 - \frac{1}{2 \cdot 4}x^4 + \ldots + (-1)^{n+1}\frac{1 \cdot 3 \cdot \ldots \cdot (2n-3)}{2 \cdot 4 \cdot \ldots \cdot (2n-2) \cdot 2n}x^{2n} + \ldots$.
8.10. $2 - 2\frac{1}{3}\left(\frac{x}{2}\right)^3 - 2\frac{2}{3^2 \cdot 2!}\left(\frac{x}{2}\right)^6 - \ldots - 2\frac{2 \cdot 5 \cdot \ldots \cdot (3n-4)}{3^n \cdot n!}\left(\frac{x}{2}\right)^{3n} - \ldots$.
8.11. $1 - \frac{1}{3!}x^2 + \frac{1}{5!}x^4 - \ldots + (-1)^{n+1}\frac{1}{(2n-1)!}x^{2n-2} + \ldots$.
8.12. $1 + \frac{1}{3!}x^6 + \frac{1}{5!}x^{12} + \ldots + \frac{1}{(2n-1)!}x^{6(n-1)} + \ldots$. **8.13.** $\ln 2 + \frac{1}{2}x + \frac{1}{8}x^2 - \frac{1}{192}x^4 + \ldots$.
8.14. $1 - \frac{1}{2}x^2 + \frac{1}{2}x^3 - \frac{5}{12}x^4 + \ldots$. **8.15.** $e\left(1 - \frac{1}{2}x^2 + \frac{1}{6}x^4 + \ldots\right)$.

8.16. $1-\frac{n}{2}x^2+\frac{3n^2-2n}{24}x^4+\ldots$. **8.17.** $\frac{1}{2}x^2+\frac{1}{12}x^4+\ldots$. **8.18.** $1+x^2-\frac{1}{2}x^3+\frac{5}{6}x^4+\ldots$. **8.19.** $x^3/6$. **8.20.** $x^4/4$. **8.21.** $8x^3/3$. **8.22.** x^2. **8.23.** $x^4/8$. **8.24.** $4x^3$. **8.25.** $x^7/30$. **8.26.** $x^6/12$.

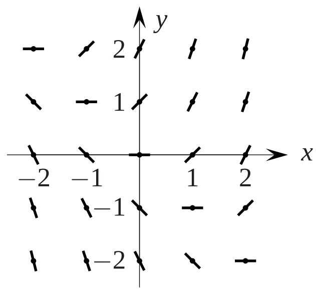
Chapter 9.

9.1. $\frac{1}{5}x^5+C$. **9.2.** $\sqrt{x}+C$. **9.3.** $\frac{5}{4}x^4+2x^3-\frac{3}{2}x^2+x+C$.
9.4. $\frac{300}{65}x^{0.65}+C$. **9.5.** $\frac{1}{32}x^4+\frac{1}{4}x^3+\frac{3}{4}x^2+x+C$. **9.6.** $\frac{2}{x}+C$.
9.7. $\frac{5^x}{\ln 5}+C$. **9.8.** $\frac{3}{14}x^{14/3}+\frac{3}{2}x^{8/3}+6x^{2/3}+C$. **9.9.** $\frac{1}{2\sqrt{6}}\ln\frac{x-\sqrt{1.5}}{x+\sqrt{1.5}}+C$.
9.10. $\frac{1}{2}x^2+\frac{12}{7}x^{7/6}+3x^{1/3}+C$. **9.11.** $2x+4\ln|x|+C$.
9.12. $x-2\tan^{-1}\dfrac{x}{2}+C$. **9.13.** $\frac{x^2}{6}-\frac{1}{3}x+C$. **9.14.** $\frac{x^4}{4}-\frac{x^2}{2}+\tan^{-1}x+C$.
9.15. $\frac{2}{5}x^{5/2}-x^2+\frac{8}{3}x^{3/2}-8x+C$. **9.16.** $\frac{x-\sin x}{2}+C$. **9.17.** $\frac{x+\sin x}{2}+C$.
9.18. $-2\cos x+C$. **9.19.** $\tan x-x+C$. **9.20.** $-\cot x-x+C$.
9.21. $\tan x+C$. **9.22.** $-\cot x+C$. **9.23.** $-\frac{5^{-x}}{\ln 5}+\frac{1}{\ln\frac{3}{5}}\left(\frac{3}{5}\right)^x+C$. **9.24.** $\frac{\pi}{2}x+C$.
9.25. $\frac{\pi}{2}x+C$. **9.26.** $-2e^{1-\sqrt{x}}+C$. **9.27.** $-\frac{1}{2}\sqrt{2x+1}+\frac{1}{6}\sqrt{(2x+1)^3}+C$.
9.28. $e^{x^2+x}+C$. **9.29.** $\ln(x^2+3x+10)+C$. **9.30.** $\ln|x^3-x^2-5x+7|+C$.
9.31. $\frac{2}{7}(x-1)^{7/2}+\frac{6}{5}(x-1)^{5/2}+2(x-1)^{3/2}+2(x-1)^{1/2}+C$.
9.32. $\frac{1}{2}x-\frac{3}{4}\ln|2x+3|+C$. **9.33.** $-\frac{11}{x-2}+4\ln|x-2|+C$. **9.34.** $\frac{1}{5}\cos^5 x-\frac{1}{3}\cos^3 x+C$.
9.35. $\tan x+C$. **9.36.** $-\frac{1}{\ln 2}2^{\cos x}+C$. **9.37.** $\frac{1}{24}\ln\left|\frac{4+3x}{4-3x}\right|+C$. **9.38.** $\ln\left|\frac{\sqrt{x+1}-1}{\sqrt{x+1}+1}\right|+C$.
9.39. $-\frac{1}{2}\cot(x^2-1)+C$. **9.40.** $2\sqrt{\tan x}+C$. **9.41.** $3\tan x-\frac{1}{2\cos^2 x}+C$.
9.42. $-\frac{1}{2(\sin^{-1}x)^2}+C$. **9.43.** $\frac{1}{2}\tan^{-1}(\frac{1}{2}e^x)+C$. **9.44.** $\frac{1}{4}\sin^{-1}(x^4)+C$.
9.45. $-(6+\frac{x^2}{3})\sqrt{9-x^2}+C$. **9.46.** $\ln|x^2-4x+10|+C$.
9.47. $\cos\left(\frac{1}{x}+5\right)+C$. **9.48.** $\frac{-1}{5\ln^5 x}+C$. **9.49.** $-\frac{1}{6}\cos^6(2x)+C$.
9.50. $\frac{1}{10\ln 3}3^{5x^2+4}+C$. **9.51.** $-\frac{2}{5}\sqrt{\cos^{-1}5x}+C$. **9.52.** $(\tan^{-1}x)^3/3+C$.
9.53. $\frac{3}{2}\ln(x^2+9)-\frac{1}{3}\tan^{-1}(\frac{x}{3})+C$. **9.54.** $2\sqrt{x-2}+\sqrt{2}\tan^{-1}\sqrt{\frac{x-2}{2}}+C$.
9.55. $-2\sqrt{1-x^2}-\frac{2}{3}(\sin^{-1}x)^{3/2}+C$. **9.56.** $\frac{1}{\ln 3}3^{\tan x}+C$.
9.57. $2\sqrt{x}-3\sqrt[3]{x}+6\sqrt[6]{x}-6\ln(\sqrt[6]{x}+1)+C$. **9.58.** $x-4\sqrt{x}+8\ln(\sqrt{x}+2)+C$.
9.59. $\dfrac{4}{5}\sqrt[4]{x^5}-x+\dfrac{4}{3}\sqrt[4]{x^3}-2\sqrt{x}+4\sqrt[4]{x}-4\ln(\sqrt[4]{x}+1)+C$. **9.60.** $\sin x-x\cos x+C$.
9.61. $x^2\sin x-2\sin x+2x\cos x+C$. **9.62.** $-\frac{1}{2}x^2\cos(2x)+\frac{1}{4}\cos(2x)+\frac{1}{2}x\sin(2x)+C$.
9.63. $-(x+1)e^{-x}+C$. **9.64.** $(x^2-2x+2)e^x+C$.
9.65. $-\frac{1}{3}x^2e^{-3x}-\frac{2}{9}xe^{-3x}-\frac{2}{27}e^{-3x}+C$. **9.66.** $-x\cot x+\ln|\sin x|+C$.
9.67. $-\frac{\ln x}{2x^2}-\frac{1}{4x^2}+C$. **9.68.** $x(\ln x-1)+C$. **9.69.** $x\cos^{-1}3x-\frac{1}{3}\sqrt{1-9x^2}+C$.
9.70. $\frac{1}{2}(x\sqrt{1-x^2}+\sin^{-1}x)+C$. **9.71.** $\frac{x}{2}\sqrt{x^2+4}+2\ln\left|x+\sqrt{x^2+4}\right|+C$.
9.72. $\frac{x^3}{9}(3\ln(2x)-1)+C$. **9.73.** $\frac{(x+1)^2}{4}(2\ln(x+1)-1)+C$.
9.74. $\frac{x}{8(x^2+4)}+\frac{1}{16}\tan^{-1}\frac{x}{2}+C$. **9.75.** $\frac{x^2}{4}\sqrt{1-x^4}+\frac{1}{4}\sin^{-1}(x^2)+C$.
9.76. $\frac{3^{-x}}{\ln^3 3}(-x^2\ln^2 3+x\ln^2 3-2x\ln 3+\ln 3-2)+C$.
9.77. $x(\tan^{-1}x-\cot^{-1}x)-\ln(1+x^2)+C$.
9.78. $\ln(x^2+x+1)(x+\frac{1}{2})-2x+\sqrt{3}\tan^{-1}\frac{2x+1}{\sqrt{3}}+C$.
9.79. $x\tan^{-1}(x^2)-\frac{\sqrt{2}}{4}\ln\frac{x^2-\sqrt{2}x+1}{x^2+\sqrt{2}x+1}-\frac{\sqrt{2}}{2}(\tan^{-1}(\sqrt{2}x+1)-\tan^{-1}(\sqrt{2}x-1))+C$.

9.80. $\ln|x-5|-\ln|x-4|+C$. **9.81.** $15\ln|x-5|-13\ln|x-4|+C$.
9.82. $x+21\ln|x-5|-13\ln|x-4|+C$.
9.83. $\frac{x^2}{2}+9x+125\ln|x-5|-64\ln|x-4|+C$.
9.84. $x^3+x^2-5x+18\ln|x+3|+C$. **9.85.** $6x+2\ln|2x-1|-5\ln|x+1|+C$.
9.86. $\frac{13}{15}\ln|x-1|-\frac{2}{3}\ln|x+2|-\frac{1}{5}\ln|x+4|+C$. **9.87.** $\frac{11}{x+3}+3\ln|x+3|+C$.
9.88. $x-3\ln|x+2|+C$. **9.89.** $-\frac{1}{2(x-1)}+\frac{1}{4}\ln\left|\frac{x+1}{x-1}\right|+C$.
9.90. $-\frac{1}{x+2}-\frac{1}{x+3}+2\ln\left|\frac{x+3}{x+2}\right|+C$. **9.91.** $-\frac{16}{x-2}-\frac{2}{3}\ln|x-2|+\frac{5}{3}\ln|x+1|+C$.
9.92. $x+3\ln|x-1|+C$. **9.93.** $\ln|x-1|-\frac{1}{2}\ln(x^2+1)+\tan^{-1}x+C$.
9.94. $\frac{3}{2}\tan^{-1}\frac{x+1}{2}+C$. **9.95.** $4\ln|x+2|-\frac{3}{2}\ln(x^2+2x+2)+\tan^{-1}(x+1)+C$.
9.96. $\frac{1}{10}\ln\frac{x^2+4}{x^2+9}+\frac{3}{2}\tan^{-1}\frac{x}{2}-\tan^{-1}\frac{x}{3}+C$. **9.97.** $\frac{1}{2}\ln(x^2+1)-\ln|x+1|-\frac{3}{x+1}+C$.

Chapter 10.

10.5. 5 and 7. **10.6.** $y_0<-1$. **10.7.**



. **10.8.** The solutions have inflection points; $\lim\limits_{x\to\infty}y(x)=\infty$ and $\lim\limits_{x\to-\infty}y(x)=-\infty$.
10.9. a) $-2<y_0<0$ and $y_0>2$; b) $y_0<-2$ and $0<y_0<2$; c) $-2\le y_0\le 2$;

d) $\lim\limits_{x\to\infty}y(x)=\begin{cases}-\infty, & y_0<-2;\\ -2, & y_0=-2;\\ 0, & -2<y_0<2;\\ 2, & y_0=2;\\ \infty, & y_0>2.\end{cases}$. **10.10.** $y=\frac{x^2}{2}+C$. **10.11.** $y^3=\frac{x^3}{3}+C$.

10.12. $y^2=\frac{x}{Cx+2}$. **10.13.** $y^{3/2}=\frac{3}{2}\sqrt{x}+C$. **10.14.** $y^2=4x^{3/2}+C$.
10.15. $y=Ce^{2\tan^{-1}x}-\frac{1}{2}$. **10.16.** $\tan y=\frac{1}{2}\ln(1+x^2)+C$, $y=\frac{\pi}{2}+\pi n$, $n\in\mathbf{N}$.
10.17. $y=\frac{2}{\ln|x+2|+C}$, $y=0$. **10.18.** $\frac{y-x}{xy}+\ln|xy|=C$, $y=0$. **10.19.** $y=Cx^2e^{-3/x}$.
10.20. $2\sqrt{y}-\tan^{-1}x=C$, $y=0$. **10.21.** $\sin^{-1}y+\sqrt{1-x^2}=C$, $y=\pm1$.
10.22. $y=2x$. **10.23.** $y^2=x^2+8$. **10.24.** $y=e^{5-2x}$. **10.25.** $y=\sqrt[3]{3x-3x^2+19}$.
10.26. $y=2\ln|x|+\frac{x^2+5}{2}$. **10.27.** $y=e^{\sqrt{x}-2}$. **10.28.** $y=(x(\ln x-1)+1)^2$.
10.29. $\tan^{-1}y=\tan^{-1}e^x-\frac{\pi}{4}$. **10.30.** $\ln y=\sqrt{\frac{1-\cos x}{1+\cos x}}$. **10.31.** $y(x)=\pi/2$.
10.32. $y=\cos^{-1}\left(\frac{\sqrt{2}}{2}\cos x\right)$. **10.33.** $D=\frac{4}{\pi t+4}$. **10.34.** $x=t^2+t^3+4$. **10.35.** 11314.
10.36. ≈ 5.68 hours. **10.37.** ≈ 11.7 days; ≈ 2263 grams. **10.38.** a) $Q=1000/P$; b) $Q=P^2-5P+450$. **10.39.** $\approx 4.2\%$. **10.40.** ≈ 1845 years ago.
10.41. ≈ 36. **10.42.** 1.35 hours. **10.43.** ≈ 2.096 hours. **10.44.** ≈ 2.954 hours.
10.45. 4.5 hours. **10.46.** 12.375 hours. **10.47.** $\ln 0.25/\ln 0.75$; ≈ 20.53 grams.
10.48. a) $90000e^2-40000\approx \$625000$; b) $(90000e^2-40000)e/20(e-1)\approx \$49400$.
10.49. $y=x\ln|x|+Cx$. **10.50.** $y=Cx^2-5x$. **10.51.** $y^2=2x^2(C-\ln|x|)$.
10.52. $(x-y)(\ln|x|+C)=2x$ or $y=x$. **10.53.** $(y+2x)^3=C(y+x)^2$ or $y=-x$ or

$y=-2x$. **10.54.** $-\frac{x^2}{2y^2}+\ln|y|=C$ or $y=0$. **10.55.** $\ln(x^2+y^2)+2\tan^{-1}\frac{y}{x}+C=0$. **10.56.** $\sin^{-1}\left(2\frac{y}{x}\right)=C-2\ln|x|,\ x>0;\ \sin^{-1}\left(2\frac{y}{x}\right)=C+2\ln|x|,\ x<0;$ $y=\pm\frac{1}{2}x$. **10.57.** $y=1+Ce^{-x^2/2}$. **10.58.** $y=\frac{1}{3}x^2+\frac{C}{x}$. **10.59.** $y=x^3-x+\frac{C}{x^2}$. **10.60.** $y=\frac{1}{3}e^x+Ce^{-2x}$. **10.61.** $y=\frac{1}{4}x^2-\frac{1}{3}x+\frac{1}{2}+\frac{C}{x^2}$. **10.62.** $y=x-1+C\sqrt{x}$. **10.63.** $y=-\frac{24}{37}\cos 3x-\frac{4}{37}\sin 3x+Ce^{x/2}$. **10.64.** $y=(\sin x+C)\cos x$. **10.65.** $y=x+(1+x^2)\tan^{-1}x+C(1+x^2)$. **10.66.** $y=\frac{1}{\cos x}+\frac{C}{\sin x}$. **10.67.** $y=-\frac{1}{2}\cos x\cos 2x-\sin x+C\cos x$.

Chapter 11.

11.1. $\int_0^1\sqrt{x}dx$. **11.2.** 4. **11.3.** −7.5. **11.4.** $\frac{1}{a}-\frac{1}{b}$. **11.5.** $e-1$. **11.6.** $\ln 2$. **11.7.** 1. **11.8.** a) 2.75; b) 5.75; c) 3.875. **11.9.** 4.25. **11.10.** $\pi\approx 3.148$. **11.11.** a) ≈ 5.556; b) ≈ 5.249; c) ≈ 5.403; d) ≈ 5.403. **11.12.** 12. **11.13.** 13100 ft^2. **11.14.** 19. **11.15.** $2\sqrt{e}$. **11.16.** $-\frac{\ln 7}{8}$. **11.17.** $\ln\frac{\sqrt{5}}{2}$. **11.18.** $-7/200$. **11.19.** $2\sqrt{3}-\frac{4}{3}\sqrt{2}$. **11.20.** $9\sqrt[3]{3}-3$. **11.21.** $-\sqrt{2}$. **11.22.** $\pi/6$. **11.23.** $1-\frac{\sqrt{3}}{2}$. **11.24.** $\frac{\pi}{12}-\frac{\sqrt{3}}{8}$. **11.25.** $\frac{2}{3}-\frac{3\sqrt{3}}{8}$. **11.26.** $\frac{\pi}{16}-\frac{7\sqrt{3}}{64}$. **11.27.** $\frac{1192}{45}$. **11.28.** $\frac{(e^2+1)^4}{8}-2$. **11.29.** $2+2\ln 3-4\ln 2$. **11.30.** $-\ln(\sqrt{10}-3)-\ln(\sqrt{2}-1)$. **11.31.** $\sqrt{5}/3$. **11.32.** $\frac{\pi-2}{8}$. **11.33.** $\frac{\sqrt{3}}{4}-\frac{7}{36}$. **11.34.** $\frac{\ln(7/3)}{32}$. **11.35.** π. **11.36.** $\frac{\sqrt{3}}{3}\pi-\ln 2$. **11.37.** $\frac{\pi}{1+\pi^2}\left(1+\frac{\pi}{\sqrt{e}}\right)$. **11.38.** $20\ln 2-3$. **11.39.** $\frac{1}{4}+\frac{\pi}{24}-\frac{\sqrt{3}}{8}$. **11.40.** $\frac{5\sqrt{3}\pi}{72}-\frac{\ln 3}{8}$. **11.41.** $5-6\ln 2$. **11.42.** $\frac{2\sqrt{3}-1}{12}\pi-\frac{\sqrt{3}-1}{2}$. **11.43.** $\frac{\ln 5}{4}-\frac{1}{4}\ln(4+\frac{1}{e^2})+\frac{3}{2}$. **11.44.** $\frac{4\sqrt{3}-\pi}{3}$. **11.46.** a) $5/4$; b) 27; c) $a=5/4$, $b=-5/4$. **11.47.** $\frac{4\sqrt{2}-2}{3}$. **11.48.** $\ln 2/2$. **11.49.** $\sqrt{3}/4$. **11.50.** $2\sqrt{3}-\frac{2\pi}{3}$. **11.51.** $4/e$. **11.52.** The integral is positive. **11.54.** 1. **11.55.** $1/2$. **11.56.** $\sin 1/e$. **11.57.** 1. **11.58.** 6. **11.59.** $27/2$. **11.60.** $68/3$. **11.61.** e^3-1/e^3. **11.62.** 6. **11.63.** $208/3$. **11.64.** $3/2$. **11.65.** $1/4+2\ln 2$. **11.66.** $\ln 3$. **11.67.** $8\pi-2\sqrt{3}$. **11.68.** $2-\pi/2$. **11.69.** $32/3$. **11.70.** $2-\sqrt{2}$. **11.71.** $125/6$. **11.72.** $5\ln 5-4$. **11.73.** $e+\frac{1}{e}-2$. **11.74.** $\sqrt{2}-1$. **11.75.** $\ln 2/2$. **11.76.** $\sqrt{3}\pi/3-\ln 2$. **11.77.** $(e^2+3e^{-2})/4$. **11.78.** a) $\frac{1+\sqrt{3}}{2}$; b) $\cos^{-1}\frac{\sqrt{3}-1}{4}$. **11.79.** $316\pi/3$. **11.80.** $56\pi/27$. **11.81.** $32\pi/3$. **11.82.** $208\pi/15$. **11.83.** $64\pi/5$. **11.84.** a) $\frac{48\pi}{5}$; b) $\frac{24\pi}{5}$. **11.85.** $3\pi/10$. **11.86.** a) $\pi/30$; b) $\pi/6$; c) $\pi/2$; d) $5\pi/6$; e) $11\pi/30$; f) $19\pi/30$. **11.87.** $2\sqrt{3}\pi$. **11.88.** a) $\pi/2$; b) $32\pi/15$. **11.89.** $2\pi/3$. **11.90.** $e-2$. **11.91.** $9/2$. **11.92.** $4\sqrt{3}/3$. **11.93.** $512/3$. **11.94.** 5; 13. **11.95.** 40 meters; 41 meters. **11.96.** 20 meters. **11.97.** 5 m/sec^2; 3 kilometers. **11.98.** a) $v(k,t)=37/(37kt+1)$; b) $\frac{1}{37^2}\frac{1}{\text{ft}}$; c) $296/29\approx 10.207$ sec; d) $1369\ln\frac{37}{29}\approx 333.519$ ft. **11.99.** $x(t)=t^3-t^2+4t+6$. **11.100.** ≈ 1.478. **11.101.** $245/4$ meters; $35\frac{\text{m}}{\text{sec}}$. **11.102.** $30+10\sqrt{11}\approx 63.2$ seconds.

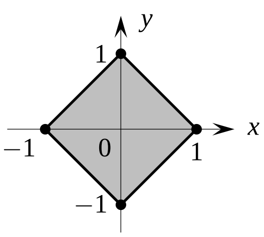
Chapter 12.

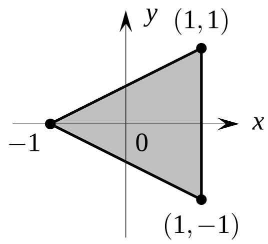
12.1. 20. **12.2.** $5(2^{4/5}-1)/4$. **12.3.** Divergent. **12.4.** π. **12.5.** $\pi/4$. **12.6.** $\pi/2$. **12.7.** 6. **12.8.** $\ln 4$. **12.9.** Divergent. **12.10.** 3. **12.11.** Divergent. **12.12.** $-4/9$. **12.13.** π. **12.14.** Divergent. **12.15.** Divergent. **12.16.** Divergent. **12.17.** $1/4$.

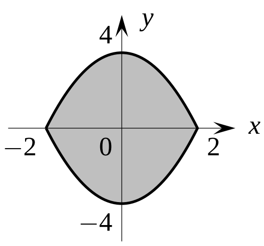
12.18. $1/50$. **12.19.** $e^5/5$. **12.20.** $1/144$. **12.21.** Divergent. **12.22.** $\pi/10$. **12.23.** $\frac{1}{3}\ln\frac{9+e^{-9}}{9}$. **12.24.** $\sqrt[3]{9}$. **12.25.** Divergent. **12.26.** $e^{-45}/9$. **12.27.** Divergent. **12.28.** $1/2$. **12.29.** $5e^4/4$. **12.30.** $(\pi-\ln 4)/4$. **12.31.** $\pi/6\sqrt{3}$. **12.32.** $\pi/4$. **12.33.** $\sqrt{\pi}/2$. **12.34.** $\sqrt{2\pi}$. **12.36.** Divergent. **12.37.** Convergent. **12.38.** Divergent. **12.39.** Convergent. **12.40.** Convergent. **12.41.** Divergent. **12.42.** Convergent. **12.43.** Divergent. **12.44.** Convergent. **12.45.** Convergent. **12.46.** Divergent. **12.47.** Convergent. **12.48.** Convergent. **12.49.** Convergent. **12.50.** Divergent. **12.51.** Convergent. **12.52.** Divergent. **12.53.** Convergent. **12.54.** Convergent. **12.55.** Divergent. **12.56.** Convergent. **12.57.** Divergent. **12.58.** Convergent. **12.59.** Divergent. **12.60.** Convergent. **12.61.** Convergent. **12.62.** Divergent. **12.63.** Convergent. **12.64.** Convergent. **12.65.** All of these integrals may be either convergent or divergent. **12.66.** i) Divergent; ii) Could be either convergent or divergent; iii) Diverges; iv) Could be either convergent or divergent. **12.67.** No.

Chapter 13.

13.1. $\{(x,y): 0\le x\le 4, 0\le y\le \frac{x}{2}\}$ or $\{(x,y): 0\le y\le 2, 2y\le x\le 4\}$.
13.2. $\{(x,y): 0\le x\le 1, x\le y\le 2-x\}$ or
$\{(x,y): 0\le y\le 1, 0\le x\le y\}\bigcup\{(x,y): 1\le y\le 2, 0\le x\le 2-y\}$.
13.3. $\left\{(x,y): -2\le x\le 2, 0\le y\le \sqrt{4-x^2}\right\}$ or
$\left\{(x,y): 0\le y\le 2, -\sqrt{4-y^2}\le x\le \sqrt{4-y^2}\right\}$.
13.4. $\{(x,y): -3\le x\le 0, 0\le y\le 2x+6\}$ or $\left\{(x,y): 0\le y\le 6, \frac{y}{2}-3\le x\le 0\right\}$.
13.5. $\left\{(x,y): -4\le x\le -3, x+4\le y\le \sqrt{x+4}\right\}$ or
$\left\{(x,y): 0\le y\le 1, y^2-4\le x\le y-4\right\}$. **13.6.** $\left\{(x,y): 0\le x\le 4, 0\le y\le \sqrt{4-\frac{x^2}{4}}\right\}$
or $\left\{(x,y): 0\le y\le 2, 0\le x\le \sqrt{16-4y^2}\right\}$.
13.7. $\left\{(x,y): -2\le x\le 1, -\frac{2}{3}x+\frac{5}{3}\le y\le \frac{x}{5}+\frac{17}{5}\right\}\bigcup\left\{(x,y): 1\le x\le 3, \frac{3}{2}x-\frac{1}{2}\le y\le \frac{x}{5}+\frac{17}{5}\right\}$
or
$\left\{(x,y): 1\le y\le 3, -\frac{3}{2}y+\frac{5}{2}\le x\le \frac{2}{3}y+\frac{1}{3}\right\}\bigcup\left\{(x,y): 3\le y\le 4, 5y-17\le x\le \frac{2}{3}y+\frac{1}{3}\right\}$.
13.8. $\left\{(x,y): 1\le x\le 3, \frac{3}{x}\le y\le 4-x\right\}$ or $\left\{(x,y): 1\le y\le 3, \frac{3}{y}\le x\le 4-y\right\}$.
13.9. $\left\{(x,y): 0\le x\le \frac{\pi}{4}, \sin x\le y\le \cos x\right\}$ or
$\left\{(x,y): 0\le y\le \frac{\sqrt{2}}{2}, 0\le x\le \sin^{-1}y\right\}\bigcup\left\{(x,y): \frac{\sqrt{2}}{2}\le y\le 1, 0\le x\le \cos^{-1}y\right\}$.
13.10. $\left\{(x,y): \frac{1}{e}\le x\le 1, \ln x\le y\le 0\right\}\bigcup\{(x,y): 1\le x\le e, 0\le y\le \ln x\}$
or $\left\{(x,y): -1\le y\le 0, \frac{1}{e}\le x\le e^y\right\}\bigcup\{(x,y): 0\le y\le 1, e^y\le x\le e\}$.

13.11. 

. **13.12.** 

. **13.13.** 

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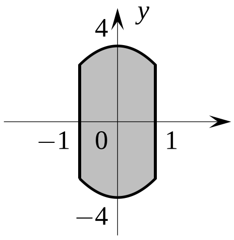
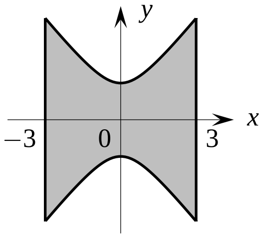
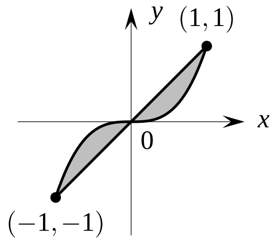
13.14. 

. **13.15.** 

. **13.16.** 

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13.17. $\int\limits_0^1 dy \int\limits_y^1 f(x,y)dx$. **13.18.** $\int\limits_0^2 dx \int\limits_0^{2x} f(x,y)dy$. **13.19.** $\int\limits_1^3 dy \int\limits_{(y+1)/2}^2 f(x,y)dx$.

13.20. $\int\limits_0^{\ln 2} dy \int\limits_{e^y}^2 f(x,y)dx$. **13.21.** $\int\limits_5^{25} dy \int\limits_{30/y}^6 f(x,y)dx + \int\limits_{25}^{30} dy \int\limits_{30/y}^{31-y} f(x,y)dx$.

13.22. $\int\limits_0^1 dx \int\limits_{1-x}^1 f(x,y)dy + \int\limits_1^2 dx \int\limits_{x-1}^1 f(x,y)dy$.

13.23. $\int\limits_{-1}^{1/2} dy \int\limits_{-y}^1 f(x,y)dx + \int\limits_{1/2}^2 dy \int\limits_{y-1}^1 f(x,y)dx - \int\limits_0^{1/2} dy \int\limits_{-1}^{y-1} f(x,y)dx - \int\limits_{1/2}^1 dy \int\limits_{-1}^{-y} f(x,y)dx$.

13.24. $3/8$. **13.25.** $4/3$. **13.26.** $\frac{3}{4}e^4 - \frac{1}{4}e^2$. **13.27.** $\frac{1}{2}e^4 - 2e$. **13.28.** 55/156.
13.29. -935/3. **13.30.** $\frac{1}{12}\left(e^{729}-1\right)$. **13.31.** $\frac{1}{6}\left(17^{3/2}-1\right)$.