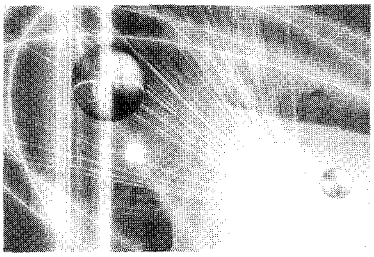


CHAPTER 14

Multiple Integration

Introduction In this chapter we extend the concept of the definite integral to functions of several variables. Defined as limits of Riemann sums, like the one-dimensional definite integral, such multiple integrals can be evaluated using successive single definite integrals. They are used to represent and calculate quantities specified in terms of densities in regions of the plane or spaces of higher dimension. In the simplest instance, the volume of a three-dimensional region is given by a *double integral* of its height over the two-dimensional plane region that is its base.

14.1 Double Integrals

The definition of the definite integral, $\int_a^b f(x)\,dx$, is motivated by the *standard area problem*, namely, the problem of finding the area of the plane region bounded by the curve $y = f(x)$, the x-axis, and the lines $x = a$ and $x = b$. Similarly, we can motivate the double integral of a function of two variables over a domain D in the plane by means of the *standard volume problem* of finding the volume of the three-dimensional region S bounded by the surface $z = f(x, y)$, the xy-plane, and the cylinder parallel to the z-axis passing through the boundary of D. (See Figure 14.1. D is called the **domain of integration**.) We will call such a three-dimensional region S a "solid," although we are not implying that it is filled with any particular substance. We will define the double integral of $f(x, y)$ over the domain D,

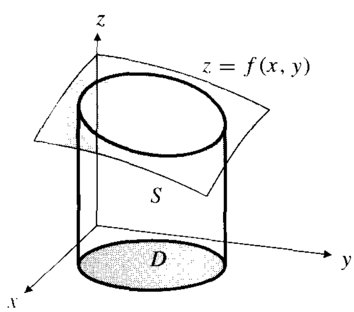


Figure 14.1 A *solid* region S lying above domain D in the xy-plane and below the surface $z = f(x, y)$

$$\iint_D f(x, y)\,dA,$$

in such a way that its value will give the volume of the solid S whenever D is a "reasonable" domain and f is a "reasonable" function with positive values.

Let us start with the case where D is a closed rectangle with sides parallel to the coordinate axes in the xy-plane, and f is a bounded function on D. If D consists of the points (x, y) such that $a \le x \le b$ and $c \le y \le d$, we can form a **partition** P of D into small rectangles by partitioning each of the intervals $[a, b]$ and $[c, d]$, say by points

$$a = x_0 < x_1 < x_2 < \cdots < x_{m-1} < x_m = b,$$
$$c = y_0 < y_1 < y_2 < \cdots < y_{n-1} < y_n = d.$$

The partition P of D then consists of the mn rectangles R_{ij} $(1 \le i \le m,\ 1 \le j \le n)$, consisting of points (x, y) for which $x_{i-1} \le x \le x_i$ and $y_{j-1} \le y \le y_j$. (See Figure 14.2.)

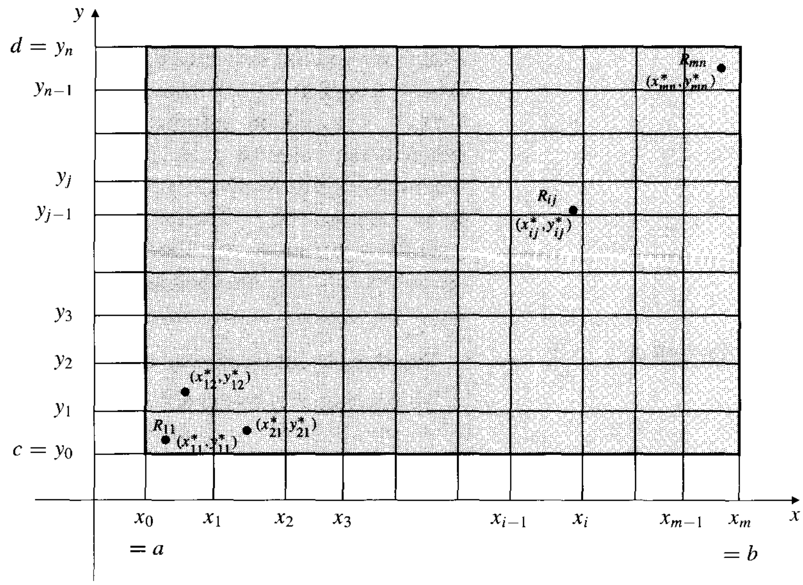


Figure 14.2 A partition of D (the large shaded rectangle) into smaller rectangles R_{ij} $(1 \le i \le m,\ 1 \le j \le n)$

The rectangle R_{ij} has area

$$\Delta A_{ij} = \Delta x_i \Delta y_j = (x_i - x_{i-1})(y_j - y_{j-1})$$

and *diameter* (i.e., diagonal length)

$$\operatorname{diam}(R_{ij}) = \sqrt{(\Delta x_i)^2 + (\Delta y_j)^2} = \sqrt{(x_i - x_{i-1})^2 + (y_j - y_{j-1})^2}.$$

The **norm** of the partition P is the largest of these subrectangle diameters:

$$\|P\| = \max_{\substack{1 \le i \le m \\ 1 \le j \le n}} \operatorname{diam}(R_{ij}).$$

Now we pick an arbitrary point (x^*_{ij}, y^*_{ij}) in each of the rectangles R_{ij} and form the **Riemann sum**

$$R(f, P) = \sum_{i=1}^{m} \sum_{j=1}^{n} f(x^*_{ij}, y^*_{ij})\, \Delta A_{ij},$$

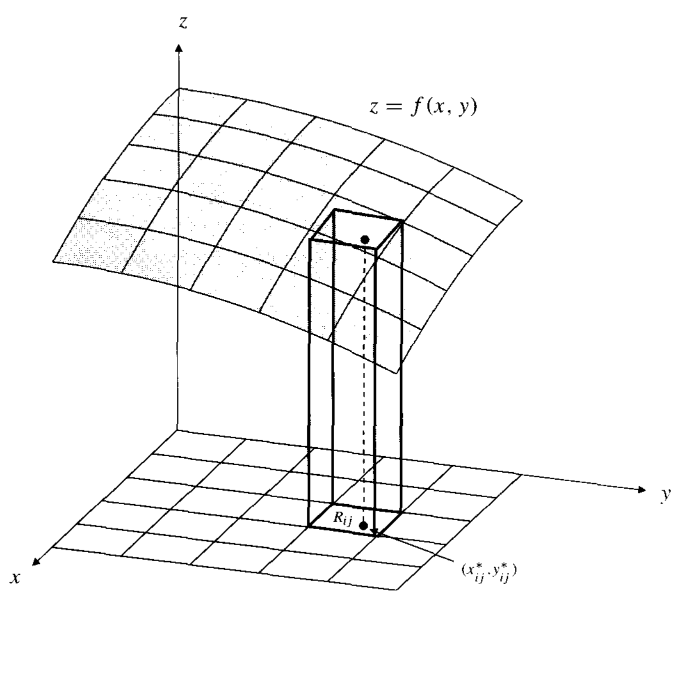


Figure 14.3 A rectangular box above rectangle R_{ij}. The Riemann sum is a sum of volumes of such boxes

which is the sum of mn terms, one for each rectangle in the partition. (Here, the *double summation* indicates the sum as i goes from 1 to m of terms, each of which is itself a sum as j goes from 1 to n.) The term corresponding to rectangle R_{ij} is, if $f(x_{ij}^*, y_{ij}^*) \geq 0$, the volume of the rectangular box whose base is R_{ij} and whose height is the value of f at (x_{ij}^*, y_{ij}^*). (See Figure 14.3.) Therefore, for positive functions f, the Riemann sum $R(f, P)$ approximates the volume above D and under the graph of f. The double integral of f over D is defined to be the limit of such Riemann sums, provided the limit exists as $\|P\| \to 0$ independently of how the points (x_{ij}^*, y_{ij}^*) are chosen. We make this precise in the following definition.

DEFINITION 1

The double integral over a rectangle

We say that f is **integrable** over the rectangle D and has **double integral**

$$I = \iint_D f(x, y)\,dA,$$

if for every positive number ϵ there exists a number δ depending on ϵ, such that

$$|R(f, P) - I| < \epsilon$$

holds for every partition P of D satisfying $\|P\| < \delta$ and for all choices of the points (x_{ij}^*, y_{ij}^*) in the subrectangles of P.

The "dA" that appears in the expression for the double integral is an *area element*. It represents the limit of the $\Delta A = \Delta x\,\Delta y$ in the Riemann sum and can also be written $dx\,dy$ or $dy\,dx$, the order being unimportant. When we evaluate double integrals by *iteration* in the next section, dA will be replaced with a product of differentials dx and dy, and the order will be important.

As is true for functions of one variable, functions that are continuous on D are integrable on D. Of course, many bounded but discontinuous functions are also integrable, but an exact description of the class of integrable functions is beyond the scope of this text.

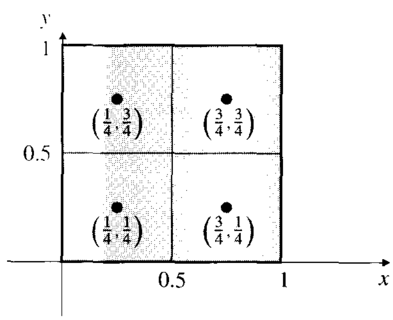


Figure 14.4 The partitioned square of Example 1

Example 1 Let D be the square $0 \leq x \leq 1$, $0 \leq y \leq 1$. Use a Riemann sum corresponding to the partition of D into four smaller squares with points selected at the centre of each to find an approximate value for

$$\iint_D (x^2 + y)\,dA.$$

Solution The required partition P is formed by the lines $x = 1/2$ and $y = 1/2$, which divide D into four squares, each of area $\Delta A = 1/4$. The centres of these squares are the points $(\frac{1}{4}, \frac{1}{4})$, $(\frac{1}{4}, \frac{3}{4})$, $(\frac{3}{4}, \frac{1}{4})$, and $(\frac{3}{4}, \frac{3}{4})$. (See Figure 14.4.) Therefore, the required approximation is

$$\iint_D (x^2 + y)\,dA \approx R(x^2 + y, P) = \left(\frac{1}{16} + \frac{1}{4}\right)\frac{1}{4} + \left(\frac{1}{16} + \frac{3}{4}\right)\frac{1}{4} + \left(\frac{9}{16} + \frac{1}{4}\right)\frac{1}{4} + \left(\frac{9}{16} + \frac{3}{4}\right)\frac{1}{4} = \frac{13}{16} = 0.8125.$$

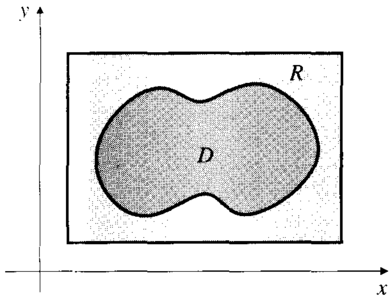


Figure 14.5

Double Integrals over More General Domains

It is often necessary to use double integrals of bounded functions $f(x, y)$ over domains that are not rectangles. If the domain D is *bounded*, we can choose a rectangle R with sides parallel to the coordinate axes such that D is contained inside R. (See Figure 14.5.) If $f(x, y)$ is defined on D, we can extend its domain to be R by defining $f(x, y) = 0$ for points in R that are outside of D. The integral of f over D can then be defined to be the integral of the extended function over the rectangle R.

DEFINITION 2

If $f(x, y)$ is defined and bounded on domain D, let $\hat{f}$ be the extension of f that is zero everywhere outside D:

$$\hat{f}(x, y) = \begin{cases} f(x, y), & \text{if } (x, y) \text{ belongs to } D \\ 0, & \text{if } (x, y) \text{ does not belong to } D. \end{cases}$$

If D is a *bounded* domain, then it is contained in some rectangle R with sides parallel to the coordinate axes. We say that f is **integrable** over D and define the **double integral** of f over D to be

$$\iint_D f(x, y)\,dA = \iint_R \hat{f}(x, y)\,dA,$$

provided that $\hat{f}$ is integrable over R.

This definition makes sense because the values of $\hat{f}$ in the part of R outside of D are all zero, so do not contribute anything to the value of the integral. However, even if f is continuous on D, $\hat{f}$ will not be continuous on R unless $f(x, y) \to 0$ as (x, y) approaches the boundary of D. Nevertheless, if f and D are "well-behaved," the integral will exist. We cannot delve too deeply into what constitutes *well-behaved*, but assert, without proof, the following theorem that will assure us that most of the double integrals we encounter do, in fact, exist.

THEOREM 1

If f is continuous on a *closed, bounded* domain D whose boundary consists of finitely many curves of finite length, then f is integrable on D.

According to Theorem 2 of Section 13.1, a continuous function is bounded if its domain is closed and bounded. Generally, however, it is not necessary to restrict our domains to be closed. If D is a bounded domain and $\text{int}(D)$ is its interior (an open set), and if f is integrable on D, then

$$\iint_D f(x, y)\,dA = \iint_{\text{int}(D)} f(x, y)\,dA.$$

We will discuss *improper double integrals* of unbounded functions or over unbounded domains in Section 14.3.

Properties of the Double Integral

Some properties of double integrals are analogous to properties of the one-dimensional definite integral and require little comment: if f and g are integrable over D, and if L and M are constants, then

(a) $\iint_D f(x, y)\, dA = 0$ if D has zero area.

(b) **Area of a domain:** $\iint_D 1\, dA =$ area of D (because it is the volume of a cylinder with base D and height 1).

(c) **Integrals representing volumes:**
If $f(x, y) \geq 0$ on D, then $\iint_D f(x, y)\, dA = V \geq 0$, where V is the volume of the solid lying vertically above D and below the surface $z = f(x, y)$.

(d) If $f(x, y) \leq 0$ on D, then $\iint_D f(x, y)\, dA = -V \leq 0$, where V is the volume of the solid lying vertically below D and above the surface $z = f(x, y)$.

(e) **Linear dependence on the integrand:**
$\iint_D \big(Lf(x, y)+Mg(x, y)\big)\, dA = L\iint_D f(x, y)\, dA+M\iint_D g(x, y)\, dA.$

(f) **Inequalities are preserved:**
If $f(x, y) \leq g(x, y)$ on D, then $\iint_D f(x, y)\, dA \leq \iint_D g(x, y)\, dA.$

(g) **The triangle inequality:** $\left|\iint_D f(x, y)\, dA\right| \leq \iint_D |f(x, y)|\, dA.$

(h) **Additivity of domains:** If $D_1, D_2, \ldots, D_k$ are nonoverlapping domains on each of which f is integrable, then f is integrable over the union $D = D_1 \cup D_2 \cup \cdots \cup D_k$ and

$$\iint_D f(x, y)\, dA = \sum_{j=1}^{k} \iint_{D_j} f(x, y)\, dA.$$

Nonoverlapping domains can share boundary points but have no interior points in common.

Double Integrals by Inspection

As yet, we have not said anything about how to *evaluate* a double integral. The main technique for doing this, called *iteration*, will be developed in the next section, but it is worth pointing out that double integrals can sometimes be evaluated using symmetry arguments or by interpreting them as volumes that we already know.

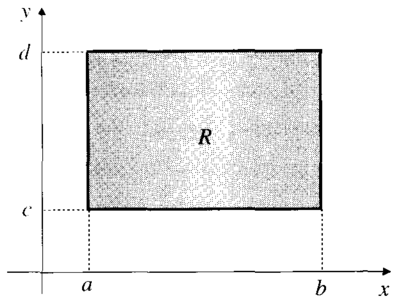


Figure 14.6

Example 2 If R is the rectangle $a \le x \le b, c \le y \le d$, then

$$\iint_R 3\,dA = 3 \times \text{area of } R = 3(b-a)(d-c).$$

Here, the integrand is $f(x, y) = 3$ and the integral is equal to the volume of the solid box of height 3 whose base is the rectangle R. (See Figure 14.6.)

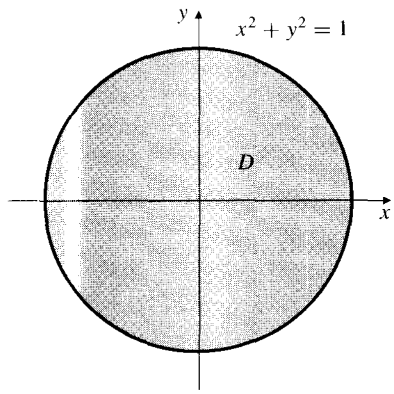


Figure 14.7

Example 3 Evaluate $I = \displaystyle\iint_{x^2+y^2\le 1} (\sin x + y^3 + 3)\,dA$.

Solution The integral can be expressed as the sum of three integrals by property (e) of double integrals:

$$\begin{aligned} I &= \iint_{x^2+y^2\le 1} \sin x\,dA + \iint_{x^2+y^2\le 1} y^3\,dA + \iint_{x^2+y^2\le 1} 3\,dA \\ &= I_1 + I_2 + I_3. \end{aligned}$$

The domain of integration (Figure 14.7) is a circular disk of radius 1 centred at the origin. Since $f(x, y) = \sin x$ is an *odd* function of x, its graph bounds as much volume below the xy-plane in the region $x < 0$ as it does above the xy-plane in the region $x > 0$. These two contributions to the double integral cancel, so $I_1 = 0$. Note that symmetry of *both* the domain *and* the integrand is necessary for this argument.

Similarly, $I_2 = 0$ because y^3 is an odd function and D is symmetric about the x-axis.

Finally,

$$I_3 = \iint_D 3\,dA = 3 \times \text{area of } D = 3\pi.$$

Thus $I = 0 + 0 + 3\pi = 3\pi$.

Example 4 If D is the disk of Example 3, the integral

$$\iint_D \sqrt{1 - x^2 - y^2}\,dA$$

represents the volume of a hemisphere of radius 1 and so has the value $2\pi/3$.

When evaluating double integrals, always be alert for situations such as those in the above examples. You can save much time by not trying to calculate an integral whose value should be obvious without calculation.

Exercises 14.1

Exercises 1–6 refer to the double integral

$$I = \iint_D (5 - x - y)\,dA,$$

where D is the rectangle $0 \le x \le 3$, $0 \le y \le 2$. P is the partition of D into six squares of side 1 as shown in Figure 14.8.

In Exercises 1–5, calculate the Riemann sums for I corresponding to the given choices of points (x_{ij}^*, y_{ij}^*).

1. (x_{ij}^*, y_{ij}^*) is the upper-left corner of each square.

2. (x_{ij}^*, y_{ij}^*) is the upper-right corner of each square.

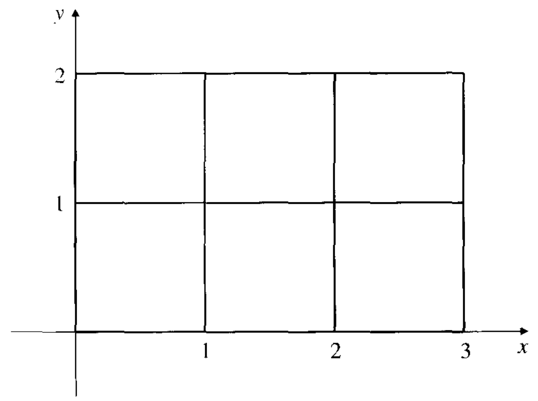


Figure 14.8

3. (x_{ij}^*, y_{ij}^*) is the lower-left corner of each square.
4. (x_{ij}^*, y_{ij}^*) is the lower-right corner of each square.
5. (x_{ij}^*, y_{ij}^*) is the centre of each square.
6. Evaluate I by interpreting it as a volume.

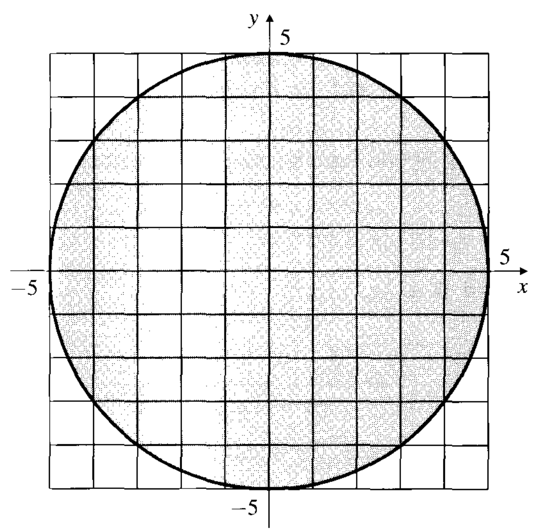


Figure 14.9

In Exercises 7–10, D is the disk $x^2 + y^2 \le 25$ and P is the partition of the square $-5 \le x \le 5$, $-5 \le y \le 5$ into one hundred 1×1 squares, as shown in Figure 14.9. Approximate the double integral

$$J = \iint_D f(x, y)\, dA,$$

where $f(x, y) = 1$ by calculating the Riemann sums $R(f, P)$ corresponding to the indicated choice of points in the small squares. *Hint:* using symmetry will make the job easier.

7. (x_{ij}^*, y_{ij}^*) is the corner of each square closest to the origin.
8. (x_{ij}^*, y_{ij}^*) is the corner of each square farthest from the origin.
9. (x_{ij}^*, y_{ij}^*) is the centre of each square.
10. Evaluate J.
11. Repeat Exercise 5 using the integrand e^x instead of $5 - x - y$.
12. Repeat Exercise 9 using $f(x, y) = x^2 + y^2$ instead of $f(x, y) = 1$.

In Exercises 13–22, evaluate the given double integral by inspection.

13. $\iint_R dA$, where R is the rectangle $-1 \le x \le 3$, $-4 \le y \le 1$
14. $\iint_D (x + 3)\, dA$, where D is the half-disk $0 \le y \le \sqrt{4 - x^2}$
15. $\iint_T (x + y)\, dA$, where T is the parallelogram having the points (2,2), $(1, -1)$, $(-2, -2)$, and $(-1, 1)$ as vertices
16. $\iint_{|x|+|y|\le 1} \left(x^3 \cos(y^2) + 3 \sin y - \pi\right) dA$
17. $\iint_{x^2+y^2\le 1} (4x^2y^3 - x + 5)\, dA$
18. $\iint_{x^2+y^2\le a^2} \sqrt{a^2 - x^2 - y^2}\, dA$
19. $\iint_{x^2+y^2\le a^2} (a - \sqrt{x^2 + y^2})\, dA$
20. $\iint_S (x + y)\, dA$, where S is the square $0 \le x \le a$, $0 \le y \le a$
21. $\iint_T (1 - x - y)\, dA$, where T is the triangle with vertices $(0, 0)$, $(1, 0)$, and $(0, 1)$
22. $\iint_R \sqrt{b^2 - y^2}\, dA$, where R is the rectangle $0 \le x \le a$, $0 \le y \le b$

14.2 Iteration of Double Integrals in Cartesian Coordinates

The existence of the double integral $\iint_D f(x, y)\, dA$ depends on f and the domain D. As we shall see, evaluation of double integrals is easiest when the domain of integration is of *simple* type.

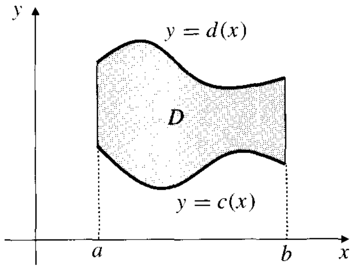


Figure 14.10 A y-simple domain

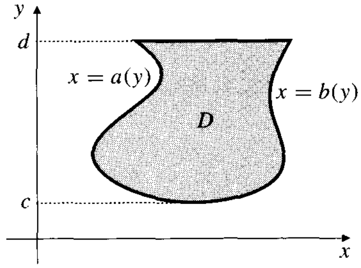


Figure 14.11 An x-simple domain

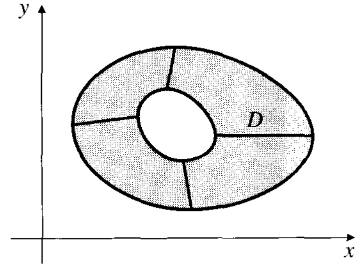


Figure 14.12 A regular domain

We say that the domain D in the xy-plane is y-**simple** if it is bounded by two vertical lines $x = a$ and $x = b$, and two continuous graphs $y = c(x)$ and $y = d(x)$ between these lines. (See Figure 14.10.) Lines parallel to the y-axis intersect a y-simple domain in an *interval* (possibly a single point) if at all. Similarly, D is x-**simple** if it is bounded by horizontal lines $y = c$ and $y = d$, and two continuous graphs $x = a(y)$ and $x = b(y)$ between these lines. (See Figure 14.11.) Many of the domains over which we will take integrals are y-simple, x-simple, or both. For example, rectangles, triangles, and disks are both x-simple and y-simple. Those domains that are neither one nor the other will usually be unions of finitely many nonoverlapping subdomains that are both x-simple and y-simple. We will call such domains **regular**. The shaded region in Figure 14.12 is divided into four subregions, each of which is both x-simple and y-simple.

It can be shown that a bounded, continuous function $f(x, y)$ is integrable over a bounded x-simple or y-simple domain and, therefore, over any regular domain.

Unlike the examples in the previous section, most double integrals cannot be evaluated by inspection. We need a technique for evaluating double integrals similar to the technique for evaluating single definite integrals in terms of antiderivatives. Since the double integral represents a volume, we can evaluate it for simple domains by a slicing technique.

Suppose, for instance, that D is y-simple and is bounded by $x = a$, $x = b$, $y = c(x)$, and $y = d(x)$, as shown in Figure 14.13(a). Then $\iint_D f(x, y)\,dA$ represents the volume of the solid region inside the vertical cylinder through the boundary of D and between the xy-plane and the surface $z = f(x, y)$. Consider the cross-section of this solid in the vertical plane perpendicular to the x-axis at position x. Note that x is constant in that plane. If we use the projections of the y- and z-axes onto the plane as coordinate axes there, the cross-section is a plane region bounded by vertical lines $y = c(x)$ and $y = d(x)$, by the horizontal line $z = 0$, and by the curve $z = f(x, y)$. The area of the cross-section is therefore given by

$$A(x) = \int_{c(x)}^{d(x)} f(x, y)\,dy.$$

The double integral $\iint_D f(x, y)\,dA$ is obtained by summing the volumes of "thin" slices of area $A(x)$ and thickness dx between $x = a$ and $x = b$ and is therefore given by

$$\iint_D f(x, y)\,dA = \int_a^b A(x)\,dx = \int_a^b \left(\int_{c(x)}^{d(x)} f(x, y)\,dy \right) dx.$$

Notationally, it is common to omit the large parentheses and write

$$\iint_D f(x, y)\, dA = \int_a^b \int_{c(x)}^{d(x)} f(x, y)\, dy\, dx,$$

or

$$\iint_D f(x, y)\, dA = \int_a^b dx \int_{c(x)}^{d(x)} f(x, y)\, dy.$$

The latter form shows more clearly which variable corresponds to which limits of integration.

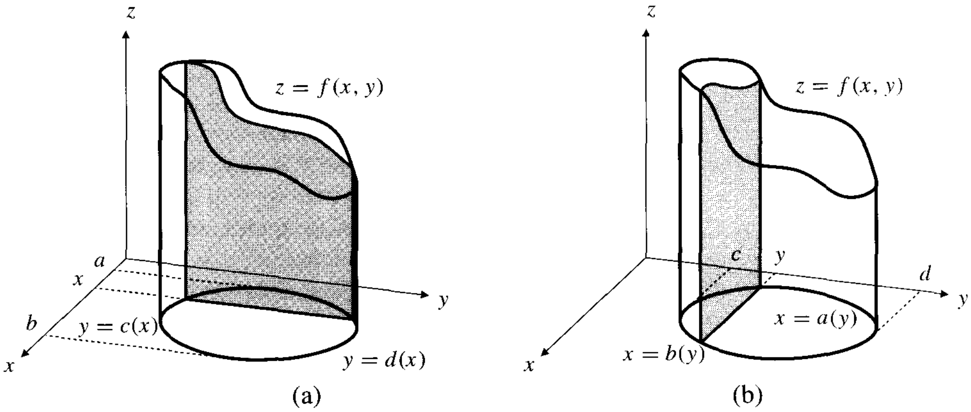


Figure 14.13

(a) In integrals over y-simple domains, slices should be perpendicular to the x-axis

(b) In integrals over x-simple domains, slices should be perpendicular to the y-axis

The expressions on the right-hand sides of the above formulas are called **iterated** integrals. **Iteration** is the process of reducing the problem of evaluating a double (or multiple) integral to one of evaluating two (or more) successive single definite integrals. In the above iteration, the integral

$$\int_{c(x)}^{d(x)} f(x, y)\, dy$$

is called the **inner** integral since it must be evaluated first. It is evaluated using standard techniques, treating x as a constant. The result of this evaluation is a function of x alone (note that both the integrand and the limits of the inner integral can depend on x) and is the integrand of the **outer** integral in which x is the variable of integration.

For double integrals over x-simple domains we can slice perpendicularly to the y-axis and obtain an iterated integral with the outer integral in the y direction. (See Figure 14.13(b).) We summarize the above discussion in the following theorem whose formal proof we will, however, not give.

THEOREM 2

Iteration of double integrals

If $f(x, y)$ is continuous on the bounded y-simple domain D given by $a \leq x \leq b$ and $c(x) \leq y \leq d(x)$, then

$$\iint_D f(x, y)\, dA = \int_a^b dx \int_{c(x)}^{d(x)} f(x, y)\, dy.$$

Similarly, if f is continuous on the x-simple domain D given by $c \leq y \leq d$ and $a(y) \leq x \leq b(y)$, then

$$\iint_D f(x, y)\, dA = \int_c^d dy \int_{a(y)}^{b(y)} f(x, y)\, dx.$$

Remark The symbol dA in the double integral is replaced in the iterated integrals by the dx and the dy. Accordingly, dA is frequently written $dx\, dy$ or $dy\, dx$ even in the double integral. The three expressions

iterated does the order of dx and dy become important. Later in this chapter we will iterate double integrals in polar coordinates, and dA will take the form $r\, dr\, d\theta$.

It is not always necessary to make a three-dimensional sketch of the solid volume represented by a double integral. In order to iterate the integral properly (in one direction or the other) it is usually sufficient to make a sketch of *the domain D* over which the integral is taken. The direction of iteration can be shown by a line along which the inner integral is taken. The following examples illustrate this.

Example 1 Find the volume of the solid lying above the square Q defined by $0 \le x \le 1$ and $1 \le y \le 2$ and below the plane $z = 4 - x - y$.

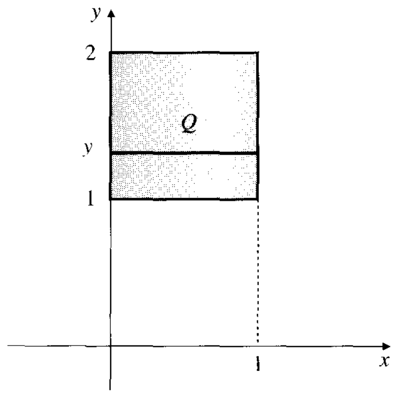


Figure 14.14 The horizontal line through Q indicates iteration with the inner integral in the x direction

Solution The square Q is both x-simple and y-simple, so the double integral giving the volume can be iterated in either direction. We will do it both ways just for practice. The horizontal line at height y in Figure 14.14 suggests that we first integrate with respect to x along this line (from 0 to 1) and then integrate the result with respect to y from 1 to 2. Iterating the double integral in this direction, we calculate

$$\begin{aligned}
\text{Volume above } Q &= \iint_Q (4 - x - y)\, dA \\
&= \int_1^2 dy \int_0^1 (4 - x - y)\, dx \\
&= \int_1^2 dy \left(4x - \frac{x^2}{2} - xy\right)\Bigg|_{x=0}^{x=1} \\
&= \int_1^2 \left(\frac{7}{2} - y\right) dy \\
&= \left(\frac{7y}{2} - \frac{y^2}{2}\right)\Bigg|_1^2 = 2 \text{ cubic units.}
\end{aligned}$$

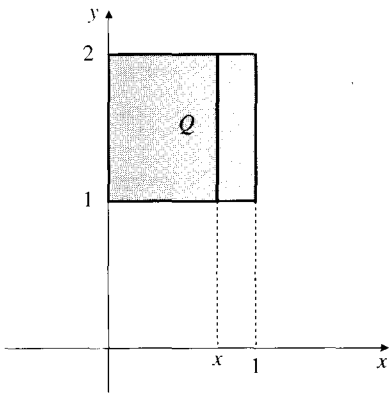


Figure 14.15 The vertical line through Q indicates iteration with the inner integral in the y direction

Using the opposite iteration, as suggested by Figure 14.15, we calculate

$$\begin{aligned}\text{Volume above } Q &= \iint_Q (4-x-y)\,dA \\ &= \int_0^1 dx \int_1^2 (4-x-y)\,dy \\ &= \int_0^1 dx \left(4y - xy - \frac{y^2}{2}\right)\bigg|_{y=1}^{y=2} \\ &= \int_0^1 \left(\frac{5}{2} - x\right) dx \\ &= \left(\frac{5x}{2} - \frac{x^2}{2}\right)\bigg|_0^1 = 2 \text{ cubic units.}\end{aligned}$$

It is comforting to get the same answer both ways! Note that because Q is a rectangle with sides parallel to the coordinate axes, the limits of the inner integrals do not depend on the variables of the outer integrals in either iteration. This cannot be expected to happen with more general domains. ■

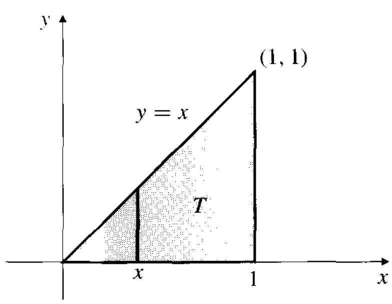


Figure 14.16 The triangular domain T with vertical line indicating iteration with inner integral in the y direction

Example 2 Evaluate $\iint_T xy\,dA$ over the triangle T with vertices $(0, 0)$, $(1, 0)$, and $(1, 1)$.

Solution The triangle T is shown in Figure 14.16. It is both x-simple and y-simple. Using the iteration corresponding to slicing in the direction shown in the figure, we obtain:

$$\begin{aligned}\iint_T xy\,dA &= \int_0^1 dx \int_0^x xy\,dy \\ &= \int_0^1 dx \left(\frac{xy^2}{2}\right)\bigg|_{y=0}^{y=x} \\ &= \int_0^1 \frac{x^3}{2}\,dx = \frac{x^4}{8}\bigg|_0^1 = \frac{1}{8}.\end{aligned}$$

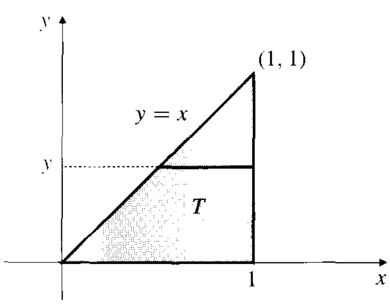


Figure 14.17 The triangular domain T with horizontal line indicating iteration with inner integral in the x direction

Iteration in the other direction (Figure 14.17) leads to the same value:

$$\begin{aligned}\iint_T xy\,dA &= \int_0^1 dy \int_y^1 xy\,dx \\ &= \int_0^1 dy \left(\frac{yx^2}{2}\right)\bigg|_{x=y}^{x=1} \\ &= \int_0^1 \frac{y}{2}(1-y^2)\,dy \\ &= \left(\frac{y^2}{4} - \frac{y^4}{8}\right)\bigg|_0^1 = \frac{1}{8}.\end{aligned}$$

■

In both of the examples above the double integral could be evaluated easily using either possible iteration. (We did them both ways just to illustrate that fact.) It often occurs, however, that a double integral is easily evaluated if iterated in one direction and very difficult, or impossible, if iterated in the other direction. Sometimes you will even encounter iterated integrals whose evaluation requires that they be expressed as double integrals and then reiterated in the opposite direction.

Example 3 Evaluate the iterated integral $I = \int_0^1 dx \int_{\sqrt{x}}^1 e^{y^3}\, dy$.

Solution We cannot antidifferentiate e^{y^3} to evaluate the inner integral in this iteration, so we express I as a double integral and identify the region over which it is taken:

$$I = \iint_D e^{y^3}\, dA,$$

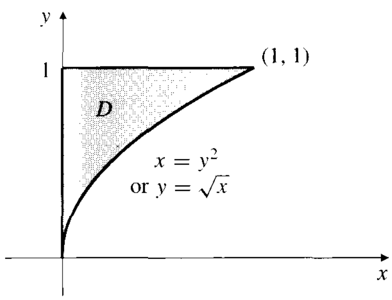


Figure 14.18 The region corresponding to the iterated integral in Example 3

where D is the region shown in Figure 14.18. Reiterating with the x integration on the inside we get

$$\begin{aligned} I &= \int_0^1 dy \int_0^{y^2} e^{y^3}\, dx \\ &= \int_0^1 e^{y^3}\, dy \int_0^{y^2} dx \\ &= \int_0^1 y^2 e^{y^3}\, dy = \left.\frac{e^{y^3}}{3}\right|_0^1 = \frac{e-1}{3}. \end{aligned}$$

■

The following is an example of the calculation of the volume of a somewhat awkward solid. Even though it is not always necessary to sketch solids to find their volumes, you are encouraged to sketch them whenever possible. When we encounter triple integrals over three-dimensional regions later in this chapter it will usually be necessary to sketch the regions. Get as much practice as you can.

Example 4 Sketch and find the volume of the solid bounded by the planes $y = 0$, $z = 0$, and $z = a - x + y$ and the parabolic cylinder $y = a - (x^2/a)$, where a is a positive constant.

Solution The solid is shown in Figure 14.19. Its base is the parabolic segment D in the xy-plane bounded by $y = 0$ and $y = a - (x^2/a)$, so the volume of the solid is given by

$$V = \iint_D (a - x + y)\, dA = \iint_D (a + y)\, dA.$$

(Note how we used symmetry to drop the x term from the integrand. This term is an odd function of x, and D is symmetric about the y-axis.) Iterating the double integral in the direction suggested by the slice shown in the figure, we obtain

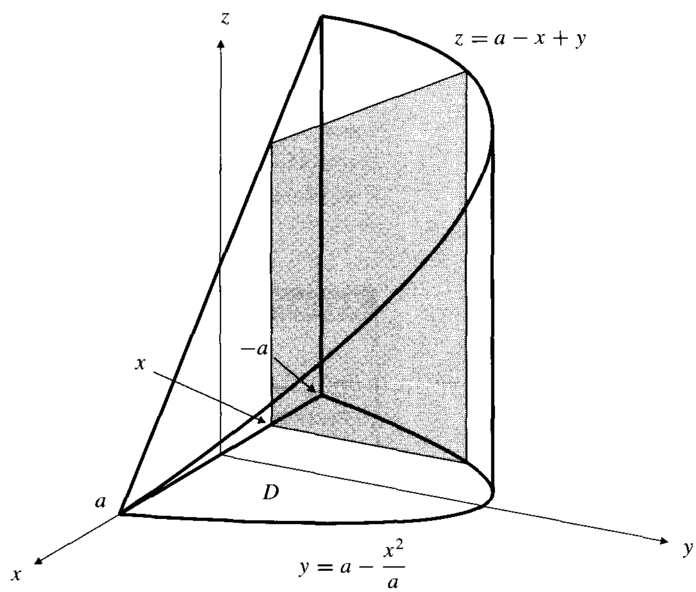


Figure 14.19 The solid in Example 4, sliced perpendicularly to the x-axis

$$
\begin{aligned}
V &= \int_{-a}^{a} dx \int_{0}^{a-(x^2/a)} (a+y)\,dy \\
&= \int_{-a}^{a} \left(ay + \frac{y^2}{2}\right)\Bigg|_{y=0}^{y=a-(x^2/a)} dx \\
&= \int_{-a}^{a} \left[a^2 - x^2 + \frac{1}{2}\left(a^2 - 2x^2 + \frac{x^4}{a^2}\right)\right] dx \\
&= 2\int_{0}^{a} \left[\frac{3}{2}a^2 - 2x^2 + \frac{x^4}{2a^2}\right] dx \\
&= \left(3a^2x - \frac{4x^3}{3} + \frac{x^5}{5a^2}\right)\Bigg|_{0}^{a} \\
&= 3a^3 - \frac{4}{3}a^3 + \frac{1}{5}a^3 = \frac{28}{15}a^3 \text{ cubic units.}
\end{aligned}
$$

Remark Maple's **int** routine can be nested to evaluate iterated double (or multiple) integrals symbolically. For instance, the iterated integral for the volume V calculated in Example 4 above can be calculated via the Maple command

```
> V = int(int(a+y, y=0..a - x^2/a), x=-a..a);
```

$$V = \frac{28}{15}a^3$$

Recall that "int" has an *inert* form "Int," which prints the integral without attempting to evaluate it symbolically. For instance, we can print an equation for the reiterated integral in the solution of Example 3 using the command

```
> Int(Int(exp(y^3),x=0..y^2),y=0..1)
        =int(int(exp(y^3),x=0..y^2),y=0..1);
```

$$\int_0^1 \int_0^{y^2} e^{(y^3)}\,dx\,dy = \frac{1}{3}e - \frac{1}{3}$$

If you want Maple to approximate an iterated integral without first trying to evaluate it symbolically, just ask it to "evalf" the inert form.

```
> evalf(Int(Int(exp(y^3),x=0..y^2),y=0..1));
```

$$.5727606095$$

Of course, Maple can't evaluate all integrals in symbolic form. If we replace $\exp(y^3)$ in the iterated integral above with $\exp(x^3)$, Maple struggles valiantly with the inner integral (of $\exp(x^3)$ with respect to x and manages to express that in terms of the gamma function and a related function of two variables called the incomplete gamma function, but it fails to evaluate the outer (y) integral.

```
> Int(Int(exp(x^3),x=0..y^2),y=0..1)
      =int(int(exp(x^3),x=0..y^2),y=0..1);
```

$$\int_0^1 \int_0^{y^2} e^{(x^3)}\,dx\,dy = \int_0^1 \frac{1}{9} \frac{y^2\left(-2\pi\sqrt{3} + 3\Gamma\left(\frac{1}{3}, -y^6\right)\Gamma\left(\frac{2}{3}\right)\right)}{\Gamma\left(\frac{2}{3}\right)(y^6)^{\left(\frac{1}{3}\right)}}\,dy$$

Again, we can force numerical approximation by using "evalf" on the inert form.

```
> Int(Int(exp(x^3),x=0..y^2),y=0..1)
      =evalf(Int(Int(exp(x^3),x=0..y^2),y=0..1));
```

$$\int_0^1 \int_0^{y^2} e^{(x^3)}\,dx\,dy = .3668032540$$

Always use the inert form when you want numerical approximation. Trying to use "evalf" with "int(int(..." can produce unexpected results such as complex values for integrals of functions that are real:

```
> evalf(int(int(exp(x^3),x=0..y^2),y=0..1));
```

$$-.1834016270 - .3176609362\,I$$

Exercises 14.2

In Exercises 1–4, calculate the given iterated integrals.

1. $\displaystyle\int_0^1 dx \int_0^x (xy + y^2)\,dy$

2. $\displaystyle\int_0^1 \int_0^y (xy + y^2)\,dx\,dy$

3. $\displaystyle\int_0^\pi \int_{-x}^x \cos y\,dy\,dx$

4. $\displaystyle\int_0^2 dy \int_0^y y^2 e^{xy}\,dx$

In Exercises 5–14, evaluate the double integrals by iteration.

5. $\displaystyle\iint_R (x^2 + y^2)\,dA$, where R is the rectangle $0 \le x \le a$, $0 \le y \le b$

6. $\displaystyle\iint_R x^2y^2\,dA$, where R is the rectangle of Exercise 5

7. $\displaystyle\iint_S (\sin x + \cos y)\,dA$, where S is the square $0 \le x \le \pi/2$, $0 \le y \le \pi/2$

8. $\displaystyle\iint_T (x - 3y)\,dA$, where T is the triangle with vertices $(0, 0)$, $(a, 0)$, and $(0, b)$

9. $\displaystyle\iint_R xy^2\,dA$, where R is the finite region in the first quadrant bounded by the curves $y = x^2$ and $x = y^2$

10. $\displaystyle\iint_D x\cos y\,dA$, where D is the finite region in the first quadrant bounded by the coordinate axes and the curve $y = 1 - x^2$

11. $\displaystyle\iint_D \ln x\,dA$, where D is the finite region in the first quadrant bounded by the line $2x + 2y = 5$ and the hyperbola $xy = 1$

12. $\displaystyle\iint_T \sqrt{a^2 - y^2}\,dA$, where T is the triangle with vertices $(0, 0)$, $(a, 0)$, and (a, a).

13. $\displaystyle\iint_R \frac{x}{y} e^y \, dA$, where R is the region $0 \le x \le 1, x^2 \le y \le x$

14. $\displaystyle\iint_T \frac{xy}{1+x^4} \, dA$, where T is the triangle with vertices $(0, 0)$, $(1, 0)$, and $(1, 1)$

In Exercises 15–18, sketch the domain of integration and evaluate the given iterated integrals.

15. $\displaystyle\int_0^1 dy \int_y^1 e^{-x^2} \, dx$

16. $\displaystyle\int_0^{\pi/2} dy \int_y^{\pi/2} \frac{\sin x}{x} \, dx$

17. $\displaystyle\int_0^1 dx \int_x^1 \frac{y^\lambda}{x^2+y^2} \, dy \quad (\lambda > 0)$

18. $\displaystyle\int_0^1 dx \int_x^{x^{1/3}} \sqrt{1-y^4} \, dy$

In Exercises 19–28, find the volumes of the indicated solids.

19. Under $z = 1 - x^2$ and above the region $0 \le x \le 1$, $0 \le y \le x$

20. Under $z = 1 - x^2$ and above the region $0 \le y \le 1$, $0 \le x \le y$

21. Under $z = 1 - x^2 - y^2$ and above the region $x \ge 0$, $y \ge 0$, $x + y \le 1$

22. Under $z = 1 - y^2$ and above $z = x^2$

23. Under the surface $z = 1/(x + y)$ and above the region in the xy-plane bounded by $x = 1$, $x = 2$, $y = 0$, and $y = x$

24. Under the surface $z = x^2 \sin(y^4)$ and above the triangle in the xy-plane with vertices $(0, 0)$, $(0, \pi^{1/4})$, and $(\pi^{1/4}, \pi^{1/4})$

25. Above the xy-plane and under the surface $z = 1 - x^2 - 2y^2$

26. Above the triangle with vertices $(0, 0)$, $(a, 0)$, and $(0, b)$ and under the plane $z = 2 - (x/a) - (y/b)$

27. Inside the two cylinders $x^2 + y^2 = a^2$ and $y^2 + z^2 = a^2$

28. Inside the cylinder $x^2 + 2y^2 = 8$, above the plane $z = y - 4$ and below the plane $z = 8 - x$

*29. Suppose that $f(x, t)$ and $f_1(x, t)$ are continuous on the rectangle $a \le x \le b$ and $c \le t \le d$. Let

$$g(x) = \int_c^d f(x, t) \, dt \quad \text{and} \quad G(x) = \int_c^d f_1(x, t) \, dt.$$

Show that $g'(x) = G(x)$ for $a < x < b$. *Hint:* evaluate $\int_a^x G(u) \, du$ by reversing the order of iteration. Then differentiate the result. This is a different version of Theorem 5 of Section 13.5.

*30. Let $F'(x) = f(x)$ and $G'(x) = g(x)$ on the interval $a \le x \le b$. Let T be the triangle with vertices (a, a), (b, a), and (b, b). By iterating $\iint_T f(x)g(y) \, dA$ in both directions, show that

$$\int_a^b f(x)G(x) \, dx$$
$$= F(b)G(b) - F(a)G(a) - \int_a^b g(y)F(y) \, dy.$$

(This is an alternative derivation of the formula for integration by parts.)

31. Use Maple's **int** routine or similar routines in other computer algebra systems to evaluate the iterated integrals in Exercises 1–4 or the iterated integrals you constructed in the remaining exercises above.

14.3 Improper Integrals and a Mean-Value Theorem

To simplify matters, the definition of the double integral given in Section 14.1 required that the domain D be bounded and that the integrand f be bounded on D. As in the single-variable case, **improper double integrals** can arise if either the domain of integration is unbounded or the integrand is unbounded near any point of the domain or its boundary.

Improper Integrals of Positive Functions

If $f(x, y) \ge 0$ on the domain D, then such an improper integral must either exist (i.e., converge to a finite value) or be infinite (diverge to infinity). Convergence or divergence of improper double integrals of such *positive* functions can be determined by iterating them and determining the convergence or divergence of any single improper integrals that result.

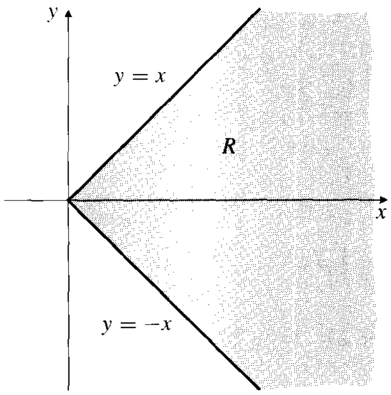


Figure 14.20 An unbounded sector of the plane

Example 1 Evaluate $I = \iint_R e^{-x^2}\,dA$. Here, R is the region where $x \ge 0$ and $-x \le y \le x$. (See Figure 14.20.)

Solution We iterate with the outer integral in the x direction:

$$\begin{aligned} I &= \int_0^\infty dx \int_{-x}^{x} e^{-x^2}\,dy \\ &= \int_0^\infty e^{-x^2}\,dx \int_{-x}^{x} dy \\ &= 2\int_0^\infty xe^{-x^2}\,dx. \end{aligned}$$

This is an improper integral that can be expressed as a limit:

$$\begin{aligned} I &= 2\lim_{r\to\infty}\int_0^r xe^{-x^2}\,dx \\ &= 2\lim_{r\to\infty}\left(-\frac{1}{2}e^{-x^2}\right)\Bigg|_0^r \\ &= \lim_{r\to\infty}(1 - e^{-r^2}) = 1. \end{aligned}$$

The given integral converges; its value is 1. ∎

Example 2 If D is the region lying above the x-axis, under the curve $y = 1/x$, and to the right of the line $x = 1$, determine whether the double integral

$$\iint_D \frac{dA}{x+y}$$

converges or diverges.

Solution The region D is sketched in Figure 14.21. We have

$$\begin{aligned} \iint_D \frac{dA}{x+y} &= \int_1^\infty dx \int_0^{1/x} \frac{dy}{x+y} \\ &= \int_1^\infty \ln(x+y)\Big|_{y=0}^{y=1/x} dx \\ &= \int_1^\infty \left(\ln\left(x+\frac{1}{x}\right) - \ln x\right) dx \\ &= \int_1^\infty \ln\left(\frac{x+\dfrac{1}{x}}{x}\right) dx = \int_1^\infty \ln\left(1+\frac{1}{x^2}\right) dx. \end{aligned}$$

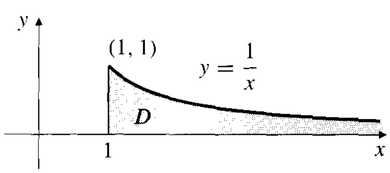


Figure 14.21 The domain of the integrand in Example 2

It happens that this integral can be evaluated exactly (see Exercise 28 below), but we are only asked to determine whether it converges, and that is more easily accomplished by estimating it. Since $0 < \ln(1+u) < u$ if $u > 0$, we have

$$0 < \iint_D \frac{dA}{x+y} < \int_1^\infty \frac{1}{x^2}\,dx = 1.$$

Therefore, the given integral converges, and its value lies between 0 and 1. ∎

Example 3 Evaluate $\iint_D \frac{1}{(x+y)^2}\,dA$, where D is the region $0 \le x \le 1$, $0 \le y \le x^2$.

Solution The integral is improper because the integrand is unbounded as (x, y) approaches $(0, 0)$, a boundary point of D. (See Figure 14.22.) Nevertheless, iteration leads to a proper integral:

$$\begin{aligned}\iint_D \frac{1}{(x+y)^2}\,dA &= \lim_{c\to 0+}\int_c^1 dx \int_0^{x^2} \frac{1}{(x+y)^2}\,dy\\ &= \lim_{c\to 0+}\int_c^1 dx \left(-\frac{1}{x+y}\right)\Bigg|_{y=0}^{y=x^2}\\ &= \lim_{c\to 0+}\int_c^1 \left(\frac{1}{x} - \frac{1}{x^2+x}\right) dx\\ &= \int_0^1 \frac{1}{x+1}\,dx = \ln(x+1)\Big|_0^1 = \ln 2.\end{aligned}$$

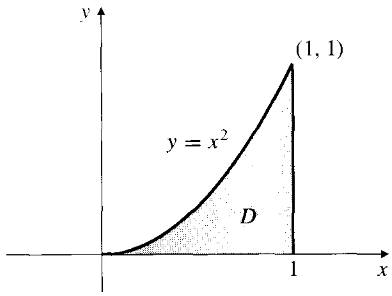


Figure 14.22 The function $\frac{1}{(x+y)^2}$ is unbounded on D

Example 4 Determine the convergence or divergence of $I = \iint_D \frac{dA}{xy}$, where D is the bounded region in the first quadrant lying between the line $y = x$ and the parabola $y = x^2$.

Solution The domain D is shown in Figure 14.23. Again, the integral is improper because the integrand $1/(xy)$ is unbounded as (x, y) approaches the boundary point $(0, 0)$. We have

$$\begin{aligned}I = \iint_D \frac{dA}{xy} &= \int_0^1 \frac{dx}{x}\int_{x^2}^{x}\frac{dy}{y}\\ &= \int_0^1 \frac{1}{x}(\ln x - \ln x^2)\,dx = -\int_0^1 \frac{\ln x}{x}\,dx.\end{aligned}$$

If we substitute $x = e^{-t}$ in this integral, we obtain

$$I = -\int_\infty^0 \frac{-t}{e^{-t}}(-e^{-t})\,dt = \int_0^\infty t\,dt,$$

which diverges to infinity.

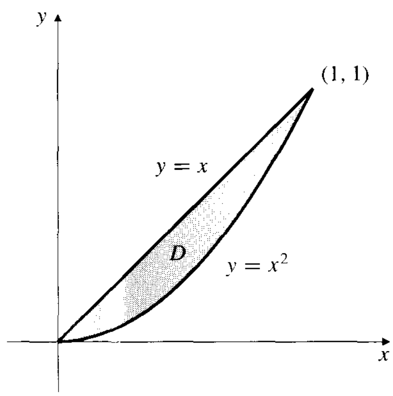


Figure 14.23 $\frac{1}{xy}$ is unbounded on the domain D

Remark In each of the examples above, the integrand was nonnegative on the domain of integration. Nonpositive integrands could have been handled similarly, but we cannot deal here with the convergence of general improper double integrals with integrands $f(x, y)$ that take both positive and negative values on the domain D of the integral. We remark, however, that such an integral cannot converge unless

$$\iint_E f(x, y)\,dA$$

is finite for every bounded, regular subdomain E of D. We cannot, in general, determine the convergence of the given integral by looking at the convergence of iterations. The double integral may diverge even if its iterations converge. (See Exercise 21 below.) In fact, opposite iterations may even give different values. This happens because of cancellation of infinite volumes of opposite sign. (Similar behaviour in one dimension is exemplified by the integral $\int_{-1}^{1} dx/x$, which does not exist, although it represents the difference between "equal" but infinite areas.) It can be shown (for a large class of functions containing, for example, continuous functions) that an improper double integral of $f(x, y)$ over D converges if the integral of $|f(x, y)|$ over D converges:

$$\iint_D |f(x, y)|\, dA \text{ converges} \quad \Rightarrow \iint_D f(x, y)\, dA \text{ converges.}$$

In this case any iterations will converge to the same value. Such double integrals are called **absolutely convergent** by analogy with absolutely convergent infinite series.

A Mean-Value Theorem for Double Integrals

Let D be a set in the xy-plane that is closed and bounded and has positive area $A = \iint_D dA$. Suppose that $f(x, y)$ is continuous on D. Then there exist points (x_1, y_1) and (x_2, y_2) in D where f assumes minimum and maximum values (see Theorem 2 of Section 13.1); that is,

$$f(x_1, y_1) \le f(x, y) \le f(x_2, y_2)$$

for all points (x, y) in D. If we integrate this inequality over D, we obtain

$$\begin{aligned} f(x_1, y_1)A &= \iint_D f(x_1, y_1)\, dA \\ &\le \iint_D f(x, y)\, dA \le \iint_D f(x_2, y_2)\, dA = f(x_2, y_2)A. \end{aligned}$$

Therefore, dividing by A, we find that the *number*

$$\bar{f} = \frac{1}{A} \iint_D f(x, y)\, dA$$

lies between the minimum and maximum values of f on D:

$$f(x_1, y_1) \le \bar{f} \le f(x_2, y_2).$$

A set D in the plane is said to be **connected** if any two points in it can be joined by a continuous parametric curve $x = x(t)$, $y = y(t)$, $(0 \le t \le 1)$, lying in D. Suppose this curve joins (x_1, y_1) (where $t = 0$) and (x_2, y_2) (where $t = 1$). Let $g(t)$ be defined by

$$g(t) = f(x(t), y(t)), \qquad 0 \le t \le 1.$$

Then g is continuous and takes the values $f(x_1, y_1)$ at $t = 0$ and $f(x_2, y_2)$ at $t = 1$. By the Intermediate-Value Theorem there exists a number t_0 between 0 and 1 such that $\bar{f} = g(t_0) = f(x_0, y_0)$, where $x_0 = x(t_0)$ and $y_0 = y(t_0)$. Thus, we have found a point (x_0, y_0) in D such that

$$\frac{1}{\text{area of } D} \iint_D f(x, y)\, dA = f(x_0, y_0).$$

We have therefore proved the following version of the Mean-Value Theorem.

THEOREM 3

A mean-malue theorem for double integrals

If the function $f(x, y)$ is continuous on a closed, bounded, connected set D in the xy-plane, then there exists a point (x_0, y_0) in D such that

$$\iint_D f(x, y)\, dA = f(x_0, y_0) \times (\text{area of } D).$$

By analogy with the definition of average value for one-variable functions, we make the following definition:

DEFINITION 3

The **average value** or **mean value** of an integrable function $f(x, y)$ over the set D is the number

$$\bar{f} = \frac{1}{\text{area of } D} \iint_D f(x, y)\, dA.$$

If $f(x, y) \geq 0$ on D, then the cylinder with base D and constant height $\bar{f}$ has volume equal to that of the solid region lying above D and below the surface $z = f(x, y)$. It is often very useful to interpret a double integral in terms of the average value of the function which is its integrand.

Example 5 The average value of x over a domain D having area A is

$$\bar{x} = \frac{1}{A} \iint_D x\, dA.$$

Of course, $\bar{x}$ is just the x-coordinate of the centroid of the region D.

Example 6 A large number of points (x, y) are chosen at random in the triangle T with vertices $(0, 0)$, $(1, 0)$, and $(1, 1)$. What is the approximate average value of $x^2 + y^2$ for these points?

Solution The approximate average value of $x^2 + y^2$ for the randomly chosen points will be the average value of that function over the triangle, namely,

$$\begin{aligned}\frac{1}{1/2} \iint_T (x^2 + y^2)\, dA &= 2 \int_0^1 dx \int_0^x (x^2 + y^2)\, dy \\ &= 2 \int_0^1 \left(x^2 y + \frac{1}{3} y^3 \right) \Bigg|_{y=0}^{y=x} dx = \frac{8}{3} \int_0^1 x^3\, dx = \frac{2}{3}.\end{aligned}$$

Example 7 Let (a, b) be an interior point of a domain D on which $f(x, y)$ is continuous. For sufficiently small positive r, the closed circular disk D_r with centre at (a, b) and radius r is contained in D. Show that

$$\lim_{r \to 0} \frac{1}{\pi r^2} \iint_{D_r} f(x, y)\, dA = f(a, b).$$

Solution If D_r is contained in D, then by Theorem 3

$$\frac{1}{\pi r^2} \iint_{D_r} f(x, y)\, dA = f(x_0, y_0)$$

for some point (x_0, y_0) in D_r. As $r \to 0$, the point (x_0, y_0) approaches (a, b). Since f is continuous at (a, b), we have $f(x_0, y_0) \to f(a, b)$. Thus,

$$\lim_{r \to 0} \frac{1}{\pi r^2} \iint_{D_r} f(x, y)\, dA = f(a, b).$$

Exercises 14.3

In Exercises 1–12, determine whether the given integral converges or diverges. Try to evaluate those that converge.

1. $\displaystyle\iint_Q e^{-x-y}\, dA$, where Q is the first quadrant of the xy-plane

2. $\displaystyle\iint_Q \frac{dA}{(1+x^2)(1+y^2)}$, where Q is the first quadrant of the xy-plane.

3. $\displaystyle\iint_S \frac{y}{1+x^2}\, dA$, where S is the strip $0 < y < 1$ in the xy-plane

4. $\displaystyle\iint_T \frac{1}{x\sqrt{y}}\, dA$ over the triangle T with vertices $(0, 0)$, $(1, 1)$, and $(1, 2)$

5. $\displaystyle\iint_Q \frac{x^2+y^2}{(1+x^2)(1+y^2)}\, dA$, where Q is the first quadrant of the xy-plane

6. $\displaystyle\iint_H \frac{1}{1+x+y}\, dA$, where H is the half-strip $0 \le x < \infty$, $0 < y < 1$

7. $\displaystyle\iint_{\mathbb{R}^2} e^{-(|x|+|y|)}\, dA$ **8.** $\displaystyle\iint_{\mathbb{R}^2} e^{-|x+y|}\, dA$

9. $\displaystyle\iint_T \frac{1}{x^3} e^{-y/x}\, dA$, where T is the region satisfying $x \ge 1$ and $0 \le y \le x$

10. $\displaystyle\iint_T \frac{dA}{x^2+y^2}$, where T is the region in Exercise 9

$*$ **11.** $\displaystyle\iint_Q e^{-xy}\, dA$, where Q is the first quadrant of the xy-plane

12. $\displaystyle\iint_R \frac{1}{x} \sin\frac{1}{x}\, dA$, where R is the region $2/\pi \le x < \infty$, $0 \le y \le 1/x$

13. Evaluate

$$I = \iint_S \frac{dA}{x+y},$$

where S is the square $0 \le x \le 1$, $0 \le y \le 1$,

(a) by direct iteration of the double integral,

(b) by using the symmetry of the integrand and the domain to write

$$I = 2\iint_T \frac{dA}{x+y},$$

where T is the triangle with vertices $(0, 0)$, $(1, 0)$, and $(1, 1)$.

14. Find the volume of the solid lying above the square S of Exercise 13 and under the surface $z = 2xy/(x^2+y^2)$.

In Exercises 15–20, a and b are given real numbers. D_k is the region $0 \le x \le 1$, $0 \le y \le x^k$, and R_k is the region $1 \le x < \infty$, $0 \le y \le x^k$. Find all real values of k for which the given integral converges.

15. $\displaystyle\iint_{D_k} \frac{dA}{x^a}$ **16.** $\displaystyle\iint_{D_k} y^b\, dA$

17. $\displaystyle\iint_{R_k} x^a\,dA$

18. $\displaystyle\iint_{R_k} \frac{dA}{y^b}$

19. $\displaystyle\iint_{D_k} x^a y^b\,dA$

20. $\displaystyle\iint_{R_k} x^a y^b\,dA$

$*$ **21.** Evaluate both iterations of the improper integral

$$\iint_S \frac{x-y}{(x+y)^3}\,dA,$$

where S is the square $0 < x < 1, 0 < y < 1$. Show that the above improper double integral does not exist, by considering

$$\iint_T \frac{x-y}{(x+y)^3}\,dA,$$

where T is that part of the square S lying under the line $x = y$.

In Exercises 22–24, find the average value of the given function over the given region.

22. x^2 over the rectangle $a \le x \le b, c \le y \le d$

23. $x^2 + y^2$ over the triangle $0 \le x \le a, 0 \le y \le a - x$

24. $1/x$ over the region $0 \le x \le 1, x^2 \le y \le \sqrt{x}$

$*$ **25.** Find the average distance from points in the quarter-disk $x^2 + y^2 \le a^2, x \ge 0, y \ge 0$, to the line $x + y = 0$.

26. Does $f(x, y) = x$ have an average value over the region $0 \le x < \infty, 0 \le y \le \dfrac{1}{1+x^2}$? If so what is it?

27. Does $f(x, y) = xy$ have an average value over the region $0 \le x < \infty, 0 \le y \le \dfrac{1}{1+x^2}$? If so what is it?

$*$ **28.** Find the exact value of the integral in Example 2. *Hint:* integrate by parts in $\int_1^\infty \ln\big(1 + (1/x^2)\big)\,dx$.

$*$ **29.** Let (a, b) be an interior point of a domain D on which the function $f(x, y)$ is continuous. For small enough $h^2 + k^2$ the rectangle R_{hk} with vertices (a, b), $(a + h, b)$, $(a, b + k)$, and $(a + h, b + k)$ is contained in D. Show that

$$\lim_{(h,k)\to(0,0)} \frac{1}{hk} \iint_{R_{hk}} f(x, y) = f(a, b).$$

Hint: see Example 7.

$*$ **30.** **(Another proof of equality of mixed partials)** Suppose that $f_{12}(x, y)$ and $f_{21}(x, y)$ are continuous in a neighbourhood of the point (a, b). Without assuming the equality of these mixed partial derivatives, show that

$$\iint_R f_{12}(x, y)\,dA = \iint_R f_{21}(x, y)\,dA,$$

where R is the rectangle with vertices (a, b), $(a + h, b)$, $(a, b + k)$, and $(a + h, b + k)$ and $h^2 + k^2$ is sufficiently small. Now use the result of Exercise 29 to show that $f_{12}(a, b) = f_{21}(a, b)$. (This reproves Theorem 1 of Section 12.4. However, in that theorem we only assumed continuity of the mixed partials *at* (a, b). Here, we assume the continuity *at all points sufficiently near* (a, b).)

14.4 Double Integrals in Polar Coordinates

For many applications of double integrals, either the domain of integration or the integrand function, or both, may be more easily expressed in terms of polar coordinates than in terms of Cartesian coordinates. Recall that a point P with Cartesian coordinates (x, y) can also be located by its polar coordinates $[r, \theta]$, where r is the distance from P to the origin O, and θ is the angle OP makes with the positive direction of the x-axis. (Positive angles θ are measured counterclockwise.) The polar and Cartesian coordinates of P are related by the transformations

$$\begin{aligned} x &= r\cos\theta, & r^2 &= x^2 + y^2,\\ y &= r\sin\theta, & \tan\theta &= y/x. \end{aligned}$$

Consider the problem of finding the volume V of the solid region lying above the xy-plane and beneath the paraboloid $z = 1 - x^2 - y^2$. Since the paraboloid intersects the xy-plane in the circle $x^2 + y^2 = 1$, the volume is given in Cartesian coordinates by

$$V = \iint_{x^2+y^2\le 1} (1 - x^2 - y^2)\, dA = \int_{-1}^{1} dx \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (1 - x^2 - y^2)\, dy.$$

Evaluating this iterated integral would require considerable effort. However, we can express the same volume in terms of polar coordinates as

$$V = \iint_{r\le 1} (1 - r^2)\, dA.$$

In order to iterate this integral, we have to know the form that the *area element* dA takes in polar coordinates.

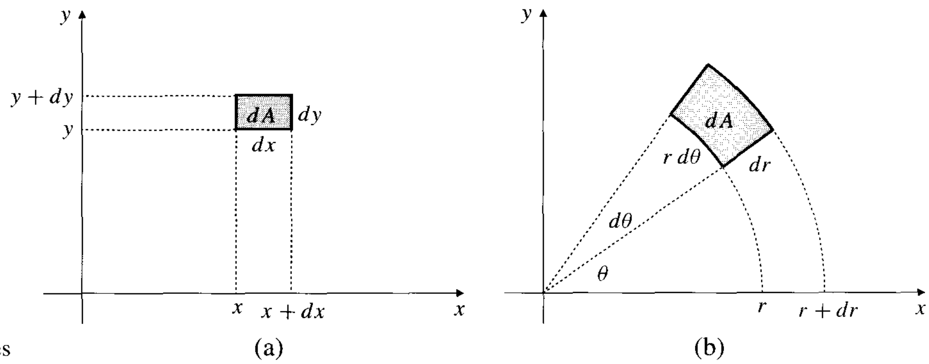


Figure 14.24
(a) $dA = dx\, dy$ in Cartesian coordinates
(b) $dA = r\, dr\, d\theta$ in polar coordinates

In the Cartesian formula for V, the area element $dA = dx\, dy$ represents the area of the "infinitesimal" region bounded by the coordinate lines at x, $x + dx$, y, and $y + dy$. (See Figure 14.24(a).) In the polar formula, the area element dA should represent the area of the "infinitesimal" region bounded by the coordinate circles with radii r and $r + dr$, and coordinate rays from the origin at angles θ and $\theta + d\theta$. (See Figure 14.24(b).) Observe that dA is approximately the area of a rectangle with dimensions dr and $r\, d\theta$. The error in this approximation becomes negligible *compared with the size of* dA as dr and $d\theta$ approach zero. Thus, in transforming a double integral between Cartesian and polar coordinates the area element transforms according to the formula

$$dx\, dy = dA = r\, dr\, d\theta.$$

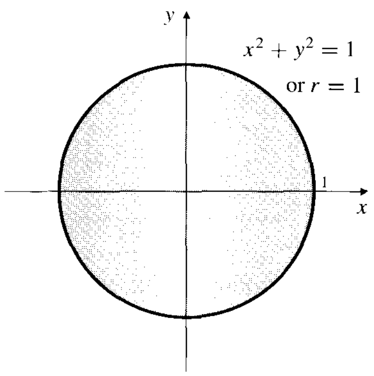


Figure 14.25 The domain in the xy-plane

In order to iterate the polar form of the double integral for V considered above, we can regard the domain of integration as a set in a plane having *Cartesian coordinates* r and θ. In the xy Cartesian plane the domain is a disk $r \le 1$ (see Figure 14.25), but in the $r\theta$ Cartesian plane (with perpendicular r- and θ-axes) the domain is the rectangle R specified by $0 \le r \le 1$ and $0 \le \theta \le 2\pi$. (See Figure 14.26.) The area element in the $r\theta$-plane is $dA^* = dr\, d\theta$, so area is not preserved under the transformation to polar coordinates ($dA = r\, dA^*$). Thus, the polar integral for V is really a Cartesian integral in the $r\theta$-plane, with integrand modified by the inclusion of an extra factor r to compensate for the change of area. It can be evaluated by standard iteration methods:

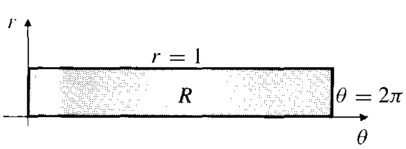


Figure 14.26 The domain in the $r\theta$-plane

$$\begin{aligned} V &= \iint_R (1 - r^2) r\, dA^* = \int_0^{2\pi} d\theta \int_0^1 (1 - r^2) r\, dr \\ &= \int_0^{2\pi} \left(\frac{r^2}{2} - \frac{r^4}{4}\right)\Bigg|_0^1 d\theta = \frac{\pi}{2} \text{ units}^3. \end{aligned}$$

Remark It is not necessary to sketch the region R in the $r\theta$-plane. We are used to thinking of polar coordinates in terms of distances and angles in the xy-plane and can easily understand from looking at the disk in Figure 14.25 that the iteration of the integral in polar coordinates corresponds to $0 \le \theta \le 2\pi$ and $0 \le r \le 1$. That is, we should be able to write the iteration

$$V = \int_0^{2\pi} d\theta \int_0^1 (1 - r^2) r\, dr$$

directly from consideration of the domain of integration in the xy-plane.

Example 1 If R is that part of the annulus $0 < a^2 \le x^2 + y^2 \le b^2$ lying in the first quadrant and below the line $y = x$, evaluate

$$I = \iint_R \frac{y^2}{x^2}\, dA.$$

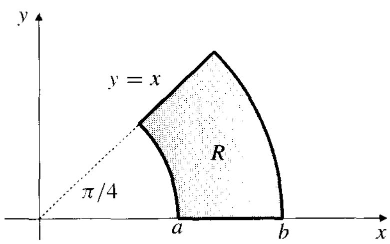


Figure 14.27

Solution Figure 14.27 shows the region R. It is specified in polar coordinates by $0 \le \theta \le \pi/4$ and $a \le r \le b$. Since

$$\frac{y^2}{x^2} = \frac{r^2 \sin^2\theta}{r^2 \cos^2\theta} = \tan^2\theta,$$

we have

$$\begin{aligned} I &= \int_0^{\pi/4} \tan^2\theta\, d\theta \int_a^b r\, dr \\ &= \frac{1}{2}(b^2 - a^2) \int_0^{\pi/4} \left(\sec^2\theta - 1\right) d\theta \\ &= \frac{1}{2}(b^2 - a^2)(\tan\theta - \theta)\Big|_0^{\pi/4} \\ &= \frac{1}{2}(b^2 - a^2)\left(1 - \frac{\pi}{4}\right) = \frac{4 - \pi}{8}(b^2 - a^2). \end{aligned}$$

■

Example 2 **(Area of a polar region)** Derive the formula for the area of the polar region R bounded by the curve $r = f(\theta)$ and the rays $\theta = \alpha$ and $\theta = \beta$. (See Figure 14.28.)

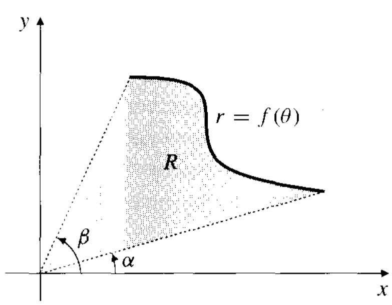


Figure 14.28 A standard area problem for polar coordinates

Solution The area A of R is numerically equal to the volume of a cylinder of height 1 above the region R:

$$\begin{aligned} A &= \iint_R dx\, dy = \iint_R r\, dr\, d\theta \\ &= \int_\alpha^\beta d\theta \int_0^{f(\theta)} r\, dr = \frac{1}{2}\int_\alpha^\beta \left(f(\theta)\right)^2 d\theta. \end{aligned}$$

Observe that the inner integral in the iteration involves integrating r along the ray specified by θ from 0 to $f(\theta)$.

■

There is no firm rule as to whether one should or should not convert a double integral from Cartesian to polar coordinates. In Example 1 above, the conversion was strongly suggested by the shape of the domain but was also indicated by the fact that the integrand, y^2/x^2, becomes a function of θ alone when converted to polar coordinates. It is usually wise to switch to polar coordinates if the switch simplifies the iteration (i.e., if the *domain* is "simpler" when expressed in terms of polar coordinates), even if the form of the integrand is made more complicated.

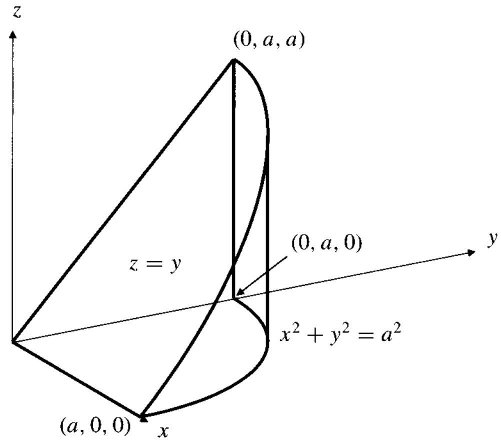


Figure 14.29 This volume is easily calculated using iteration in polar coordinates

Example 3 Find the volume of the solid lying in the first octant, inside the cylinder $x^2 + y^2 = a^2$, and under the plane $z = y$.

Solution The solid is shown in Figure 14.29. The base is a quarter disk, which is expressed in polar coordinates by the inequalities $0 \le \theta \le \pi/2$ and $0 \le r \le a$. The height is given by $z = y = r \sin\theta$. The solid has volume

$$V = \int_0^{\pi/2} d\theta \int_0^a (r \sin\theta) r\, dr = \int_0^{\pi/2} \sin\theta\, d\theta \int_0^a r^2\, dr = \frac{1}{3} a^3 \text{ units}^3.$$

The following example establishes the value of a definite integral that plays a very important role in probability theory and statistics. It is interesting that this single-variable integral cannot be evaluated by the techniques of single-variable calculus.

Example 4 **(A very important integral)** Show that

$$\int_{-\infty}^{\infty} e^{-x^2}\, dx = \sqrt{\pi}.$$

Solution The improper integral converges, and its value does not depend on what symbol we use for the variable of integration. Therefore, we can express the square of the integral as a product of two identical integrals but with their variables of integration named differently. We then interpret this product as an improper double integral and reiterate it in polar coordinates:

$$\begin{aligned}\left(\int_{-\infty}^{\infty} e^{-x^2}\,dx\right)^2 &= \int_{-\infty}^{\infty} e^{-x^2}\,dx \int_{-\infty}^{\infty} e^{-y^2}\,dy \\ &= \iint_{\mathbb{R}^2} e^{-(x^2+y^2)}\,dA \\ &= \int_0^{2\pi} d\theta \int_0^{\infty} e^{-r^2} r\,dr \\ &= 2\pi \lim_{R\to\infty} \left(-\frac{1}{2}e^{-r^2}\right)\Bigg|_0^R = \pi.\end{aligned}$$

The r integral, a convergent improper integral, was evaluated with the aid of the substitution $u = r^2$. ■

As our final example of iteration in polar coordinates let us try something a little more demanding.

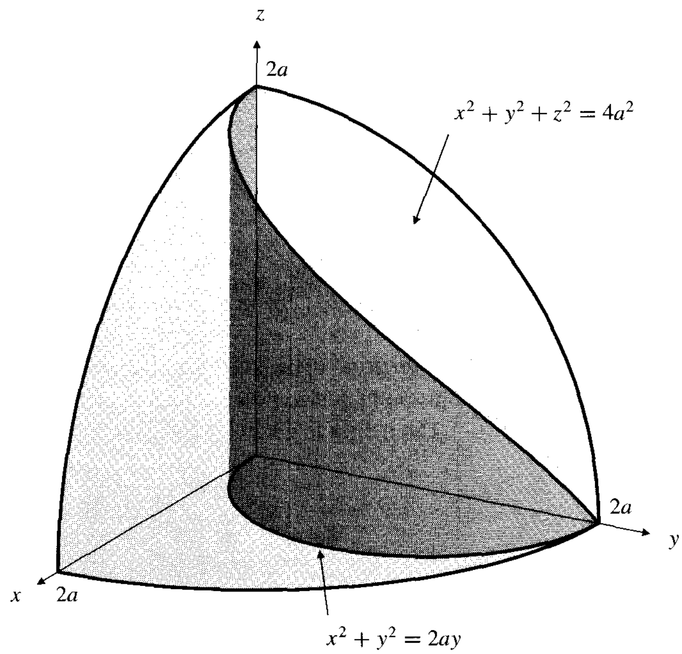


Figure 14.30 The first octant part of the intersection of the cylinder $x^2 + y^2 = 2ay$ and the sphere $x^2 + y^2 + z^2 = 4a^2$

Example 5 Find the volume of the solid region lying inside both the sphere $x^2 + y^2 + z^2 = 4a^2$ and the cylinder $x^2 + y^2 = 2ay$.

Solution The sphere is centred at the origin and has radius $2a$. The equation of the cylinder becomes

$$x^2 + (y-a)^2 = a^2$$

if we complete the square in the y terms. Thus, it is a vertical circular cylinder of radius a having its axis along the vertical line through $(0, a, 0)$. The z-axis lies on the cylinder. One-quarter of the required volume lies in the first octant. This part is shown in Figure 14.30.

If we use polar coordinates in the xy-plane, then the sphere has equation $r^2 + z^2 = 4a^2$ and the cylinder has equation $r^2 = 2ar\sin\theta$ or, more simply, $r = 2a\sin\theta$. The first octant portion of the volume lies above the region specified by the inequalities $0 \le \theta \le \pi/2$ and $0 \le r \le 2a\sin\theta$. Therefore, the total volume is

$$\begin{aligned}
V &= 4\int_0^{\pi/2} d\theta \int_0^{2a\sin\theta} \sqrt{4a^2 - r^2}\, r\, dr && \text{(Let } u = 4a^2 - r^2\text{.).}\\
&= 2\int_0^{\pi/2} d\theta \int_{4a^2\cos^2\theta}^{4a^2} \sqrt{u}\, du\\
&= \frac{4}{3}\int_0^{\pi/2} (8a^3 - 8a^3\cos^3\theta)\, d\theta && \text{(Let } v = \sin\theta\text{.).}\\
&= \frac{16}{3}\pi a^3 - \frac{32}{3}a^3 \int_0^1 (1 - v^2)\, dv\\
&= \frac{16}{3}\pi a^3 - \frac{64}{9}a^3 = \frac{16}{9}(3\pi - 4)a^3 \text{ cubic units.}
\end{aligned}$$

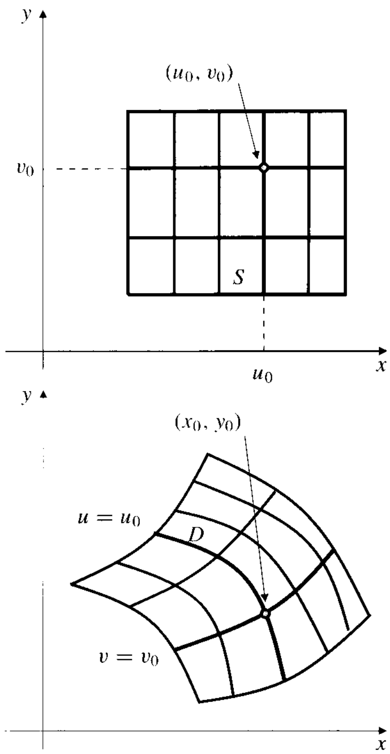


Figure 14.31 Under the transformation $\begin{cases} x = x(u, v) \\ y = y(u, v) \end{cases}$ the lines $u = u_0$ and $v = v_0$ in the uv-plane get mapped to the curves $\begin{cases} x = x(u_0, v) \\ y = y(u_0, v) \end{cases}$ and $\begin{cases} x = x(u, v_0) \\ y = y(u, v_0) \end{cases}$ in the xy-plane, which we still label as $u = u_0$ and $v = v_0$. The point (u_0, v_0) is mapped to the point (x_0, y_0)

Change of Variables in Double Integrals

The transformation of a double integral to polar coordinates is just a special case of a general change of variables formula for double integrals. Suppose that x and y are expressed as functions of two other variables u and v by the equations

$$\begin{aligned} x &= x(u, v)\\ y &= y(u, v). \end{aligned}$$

We regard these equations as defining a **transformation** (or mapping) from points (u, v) in a uv-Cartesian plane to points (x, y) in the xy-plane. (See Figure 14.31.) We say that the transformation is *one-to-one* from the set S in the uv-plane *onto* the set D in the xy-plane provided:

(i) every point in S gets mapped to a point in D,
(ii) every point in D is the image of a point in S, and
(iii) different points in S get mapped to different points in D.

If the transformation is one-to-one, the defining equations can be solved for u and v as functions of x and y, and the resulting **inverse transformation**,

$$\begin{aligned} u &= u(x, y)\\ v &= v(x, y), \end{aligned}$$

is one-to-one from D onto S.

Let us assume that the functions $x(u, v)$ and $y(u, v)$ have continuous first partial derivatives and that the Jacobian determinant

$$\frac{\partial(x, y)}{\partial(u, v)} \ne 0 \qquad \text{at} \quad (u, v).$$

As noted in Section 12.8, the Implicit Function Theorem implies that the transformation is one-to-one near (u, v) and the inverse transformation also has continuous first partial derivatives and nonzero Jacobian satisfying

$$\frac{\partial(u, v)}{\partial(x, y)} = \frac{1}{\dfrac{\partial(x, y)}{\partial(u, v)}} \quad \text{on } D.$$

Example 6 The transformation $x = r\cos\theta$, $y = r\sin\theta$ to polar coordinates has Jacobian

$$\frac{\partial(x, y)}{\partial(r, \theta)} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r.$$

Near any point except the origin (where $r = 0$) the transformation is one-to-one. (In fact, it is one-to-one from any set in the $r\theta$-plane that does not contain more than one point where $r = 0$ and lies in, say, the strip $0 \le \theta < 2\pi$.) ■

A one-to-one transformation can be used to transform the double integral

$$\iint_D f(x, y)\, dA$$

to a double integral over the corresponding set S in the uv-plane. Under the transformation, the integrand $f(x, y)$ becomes $g(u, v) = f\big(x(u, v), y(u, v)\big)$. We must discover how to express the area element $dA = dx\,dy$ in terms of the area element $du\,dv$ in the uv-plane.

For any fixed value of u (say $u = c$), the equations

$$x = x(u, v) \quad \text{and} \quad y = y(u, v)$$

define a parametric curve (with v as parameter) in the xy-plane. This curve is called a u-curve corresponding to the value $u = c$. Similarly, for fixed v the equations define a parametric curve (with parameter u) called a v-curve. Consider the differential area element bounded by the u-curves corresponding to nearby values u and $u + du$ and the v-curves corresponding to nearby values v and $v + dv$. Since these curves are smooth, for small values of du and dv the area element is approximately a parallelogram, and its area is approximately

$$dA = |\overrightarrow{PQ} \times \overrightarrow{PR}|,$$

where P, Q, and R are the points shown in Figure 14.32. The error in this approximation becomes negligible compared with dA as du and dv approach zero.

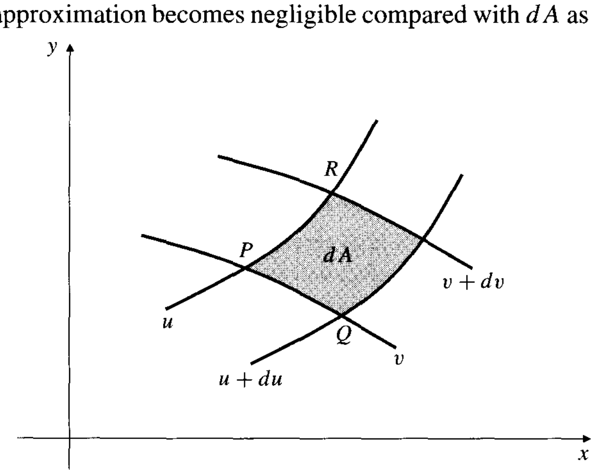


Figure 14.32 The image in the xy-plane of the area element $du\,dv$ in the uv-plane

Now $\overrightarrow{PQ} = dx\,\mathbf{i} + dy\,\mathbf{j}$, where

$$dx = \frac{\partial x}{\partial u}du + \frac{\partial x}{\partial v}dv \quad \text{and} \quad dy = \frac{\partial y}{\partial u}du + \frac{\partial y}{\partial v}dv.$$

However, $dv = 0$ along the v-curve PQ, so

$$\overrightarrow{PQ} = \frac{\partial x}{\partial u}du\,\mathbf{i} + \frac{\partial y}{\partial u}du\,\mathbf{j}.$$

Similarly,

$$\overrightarrow{PR} = \frac{\partial x}{\partial v}dv\,\mathbf{i} + \frac{\partial y}{\partial v}dv\,\mathbf{j}.$$

Hence,

$$dA = \left\| \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial x}{\partial u}du & \frac{\partial y}{\partial u}du & 0 \\ \frac{\partial x}{\partial v}dv & \frac{\partial y}{\partial v}dv & 0 \end{vmatrix} \right\| = \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\,dv;$$

that is, the absolute value of the Jacobian $\partial(x,y)/\partial(u,v)$ is the ratio between corresponding area elements in the xy-plane and the uv-plane:

$$dA = dx\,dy = \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\,dv.$$

The following theorem summarizes the change of variables procedure for a double integral.

THEOREM 4

Change of variables formula for double integrals

Let $x = x(u,v)$, $y = y(u,v)$ be a one-to-one transformation from a domain S in the uv-plane onto a domain D in the xy-plane. Suppose that the functions x and y, and their first partial derivatives with respect to u and v, are continuous in S. If $f(x,y)$ is integrable on D, and if $g(u,v) = f(x(u,v), y(u,v))$, then g is integrable on S and

$$\iint_D f(x,y)\,dx\,dy = \iint_S g(u,v) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\,dv.$$

Remark It is not necessary that S or D be closed or that the transformation be one-to-one on the boundary of S. The transformation to polar coordinates maps the rectangle $0 < r < 1$, $0 \le \theta < 2\pi$ one-to-one onto the punctured disk $0 < x^2 + y^2 < 1$ and, as in the first example in this section, we can transform an integral over the closed disk $x^2 + y^2 \le 1$ to one over the closed rectangle $0 \le r \le 1$, $0 \le \theta \le 2\pi$.

Example 7 Use an appropriate change of variables to find the area of the elliptic disk E given by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} \le 1.$$

Solution Under the transformation $x = au$, $y = bv$, the elliptic disk E is the one-to-one image of the circular disk D given by $u^2 + v^2 \le 1$. Assuming $a > 0$ and $b > 0$, we have

$$dx\,dy = \left|\frac{\partial(x, y)}{\partial(u, v)}\right| du\,dv = \left|\begin{vmatrix} a & 0 \\ 0 & b \end{vmatrix}\right| du\,dv = ab\,du\,dv.$$

Therefore, the area of E is given by

$$\iint_E 1\,dx\,dy = \iint_D ab\,du\,dv = ab \times (\text{area of } D) = \pi ab \text{ square units.}$$

It is often tempting to try to use the change of variable formula to transform the domain of a double integral into a rectangle so that iteration will be easy. As the following example shows, this usually involves defining the inverse transformation (u and v in terms of x and y). Remember that inverse transformations have reciprocal Jacobians.

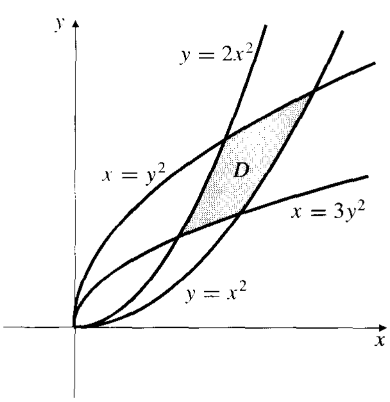


Figure 14.33

Example 8 Find the area of the finite plane region bounded by the four parabolas $y = x^2$, $y = 2x^2$, $x = y^2$, and $x = 3y^2$.

Solution The region, call it D, is sketched in Figure 14.33. Let

$$u = \frac{y}{x^2} \quad \text{and} \quad v = \frac{x}{y^2}.$$

Then the region D corresponds to the rectangle R in the uv-plane given by $1 \le u \le 2$ and $1 \le v \le 3$. Since

$$\frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} -2y/x^3 & 1/x^2 \\ 1/y^2 & -2x/y^3 \end{vmatrix} = \frac{3}{x^2y^2} = 3u^2v^2,$$

it follows that

$$\left|\frac{\partial(x, y)}{\partial(u, v)}\right| = \frac{1}{3u^2v^2}$$

and so the area of D is given by

$$\iint_D dx\,dy = \iint_R \frac{1}{3u^2v^2}\,du\,dv$$
$$= \frac{1}{3}\int_1^2 \frac{du}{u^2}\int_1^3 \frac{dv}{v^2} = \frac{1}{3} \times \frac{1}{2} \times \frac{2}{3} = \frac{1}{9} \text{ square units.}$$

The following example shows what can happen if a transformation of the domain of a double integral is not one-to-one.

Example 9 Let D be the square $0 \le x \le 1, 0 \le y \le 1$ in the xy-plane, and let S be the square $0 \le u \le 1, 0 \le v \le 1$ in the uv-plane. Show that the transformation

$$x = 4u - 4u^2, \qquad y = v$$

maps S onto D, and use it to transform the integral $I = \iint_D dx\,dy$. Compare the value of I with that of the transformed integral.

Solution Since $x = 4u - 4u^2 = 1 - (1 - 2u)^2$, the minimum value of x on the interval $0 \le u \le 1$ is 0 (at $u = 0$ and $u = 1$), and the maximum value is 1 (at $u = \frac{1}{2}$). Therefore, $x = 4u - 4u^2$ maps the interval $0 \le u \le 1$ onto the interval $0 \le x \le 1$. Since $y = v$ clearly maps $0 \le v \le 1$ onto $0 \le y \le 1$, the given transformation maps S onto D. Since

$$dx\,dy = \left|\frac{\partial(x, y)}{\partial(u, v)}\right| du\,dv = \left|\begin{vmatrix} 4 - 8u & 0 \\ 0 & 1 \end{vmatrix}\right| du\,dv = |4 - 8u|\,du\,dv,$$

transforming I leads to the integral

$$J = \iint_S |4 - 8u|\,du\,dv = 4\int_0^1 dv \int_0^1 |1 - 2u|\,du = 8\int_0^{1/2} (1 - 2u)\,du = 2.$$

However, $I = \iint_D dx\,dy =$ area of $D = 1$. The reason that $J \ne I$ is that the transformation is not one-to-one from S onto D; it actually maps S onto D twice. The rectangle R defined by $0 \le u \le \frac{1}{2}$ and $0 \le v \le 1$ is mapped one-to-one onto D by the transformation, so the appropriate transformed integral is $\iint_R |4 - 8u|\,du\,dv$, which is equal to I. ∎

Exercises 14.4

In Exercises 1–6, evaluate the given double integral over the disk D given by $x^2 + y^2 \le a^2$, where $a > 0$.

1. $\displaystyle\iint_D (x^2 + y^2)\,dA$
2. $\displaystyle\iint_D \sqrt{x^2 + y^2}\,dA$
3. $\displaystyle\iint_D \frac{1}{\sqrt{x^2 + y^2}}\,dA$
4. $\displaystyle\iint_D |x|\,dA$
5. $\displaystyle\iint_D x^2\,dA$
6. $\displaystyle\iint_D x^2y^2\,dA$

In Exercises 7–10, evaluate the given double integral over the quarter-disk Q given by $x \ge 0$, $y \ge 0$, and $x^2 + y^2 \le a^2$, where $a > 0$.

7. $\displaystyle\iint_Q y\,dA$
8. $\displaystyle\iint_Q (x + y)\,dA$
9. $\displaystyle\iint_Q e^{x^2+y^2}\,dA$
10. $\displaystyle\iint_Q \frac{2xy}{x^2 + y^2}\,dA$

11. Evaluate $\displaystyle\iint_S (x + y)\,dA$, where S is the region in the first quadrant lying inside the disk $x^2 + y^2 \le a^2$ and under the line $y = \sqrt{3}x$.
12. Find $\displaystyle\iint_S x\,dA$, where S is the disk segment $x^2 + y^2 \le 2$, $x \ge 1$.
13. Evaluate $\displaystyle\iint_T (x^2 + y^2)\,dA$, where T is the triangle with vertices $(0, 0)$, $(1, 0)$, and $(1, 1)$.
14. Evaluate $\displaystyle\iint_{x^2+y^2\le 1} \ln(x^2 + y^2)\,dA$.
15. Find the average distance from the origin to points in the disk $x^2 + y^2 \le a^2$.
16. Find the average value of $e^{-(x^2+y^2)}$ over the annular region $0 < a \le \sqrt{x^2 + y^2} \le b$.
17. For what values of k, and to what value, does the integral $\displaystyle\iint_{x^2+y^2\le 1} \frac{dA}{(x^2 + y^2)^k}$ converge?
18. For what values of k, and to what value, does the integral $\displaystyle\iint_{\mathbb{R}^2} \frac{dA}{(1 + x^2 + y^2)^k}$ converge?

19. Evaluate $\displaystyle\iint_D xy\,dA$, where D is the plane region satisfying $x \ge 0$, $0 \le y \le x$, and $x^2 + y^2 \le a^2$.

20. Evaluate $\displaystyle\iint_C y\,dA$, where C is the upper half of the cardioid disk $r \le 1 + \cos\theta$.

21. Find the volume lying between the paraboloids $z = x^2 + y^2$ and $3z = 4 - x^2 - y^2$.

22. Find the volume lying inside both the sphere $x^2 + y^2 + z^2 = a^2$ and the cylinder $x^2 + y^2 = ax$.

23. Find the volume lying inside both the sphere $x^2 + y^2 + z^2 = 2a^2$ and the cylinder $x^2 + y^2 = a^2$.

24. Find the volume of the region lying above the xy-plane, inside the cylinder $x^2 + y^2 = 4$ and below the plane $z = x + y + 4$.

$*$ **25.** Find the volume of the region lying inside all three of the circular cylinders $x^2 + y^2 = a^2$, $x^2 + z^2 = a^2$, and $y^2 + z^2 = a^2$. *Hint:* make a good sketch of the first octant part of the region and use symmetry whenever possible.

26. Find the volume of the region lying inside the circular cylinder $x^2 + y^2 = 2y$ and inside the parabolic cylinder $z^2 = y$.

$*$ **27.** Many points are chosen at random in the disk $x^2 + y^2 \le 1$. Find the approximate average value of the distance from these points to the nearest side of the smallest square that contains the disk.

$*$ **28.** Find the average value of x over the segment of the disk $x^2 + y^2 \le 4$ lying to the right of $x = 1$. What is the centroid of the segment?

29. Find the volume enclosed by the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

30. Find the volume of the region in the first octant below the paraboloid

$$z = 1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}.$$

Hint: use the change of variables $x = au$, $y = bv$.

$*$ **31.** Evaluate $\displaystyle\iint_{|x|+|y|\le a} e^{x+y}\,dA.$

32. Find $\displaystyle\iint_P (x^2 + y^2)\,dA$, where P is the parallelogram bounded by the lines $x + y = 1$, $x + y = 2$, $3x + 4y = 5$, and $3x + 4y = 6$.

33. Find the area of the region in the first quadrant bounded by the curves $xy = 1$, $xy = 4$, $y = x$, and $y = 2x$.

34. Evaluate $\displaystyle\iint_R (x^2 + y^2)\,dA$, where R is the region in the first quadrant bounded by $y = 0$, $y = x$, $xy = 1$, and $x^2 - y^2 = 1$.

$*$ **35.** Let T be the triangle with vertices $(0, 0)$, $(1, 0)$, and $(0, 1)$. Evaluate the integral $\displaystyle\iint_T e^{(y-x)/(y+x)}\,dA$,

(a) by transforming to polar coordinates

(b) by using the transformation $u = y - x$, $v = y + x$.

36. Use the method of Example 7 to find the area of the region inside the ellipse $4x^2 + 9y^2 = 36$ and above the line $2x + 3y = 6$.

$*$ **37.** **(The error function)** The error function, $\mathrm{Erf}(x)$, is defined for $x \ge 0$ by

$$\mathrm{Erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\,dt.$$

Show that $\displaystyle\left(\mathrm{Erf}(x)\right)^2 = \frac{4}{\pi}\int_0^{\pi/4}\left(1 - e^{-x^2/\cos^2\theta}\right)d\theta.$

Hence deduce that $\mathrm{Erf}(x) \ge \sqrt{1 - e^{-x^2}}$.

$*$ **38.** **(The gamma and beta functions)** The gamma function $\Gamma(x)$ and the beta function $B(x, y)$ are defined by

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\,dt, \qquad (x > 0),$$

$$B(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,dt, \qquad (x > 0,\ y > 0).$$

The gamma function satisfies

$$\Gamma(x+1) = x\Gamma(x) \qquad \text{and}$$
$$\Gamma(n+1) = n!, \qquad (n = 0, 1, 2, \ldots).$$

Deduce the following further properties of these functions:

(a) $\displaystyle\Gamma(x) = 2\int_0^\infty s^{2x-1}e^{-s^2}\,ds, \qquad (x > 0),$

(b) $\displaystyle\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}, \qquad \Gamma\left(\frac{3}{2}\right) = \frac{1}{2}\sqrt{\pi},$

(c) If $x > 0$ and $y > 0$, then

$$B(x, y) = 2\int_0^{\pi/2} \cos^{2x-1}\theta \sin^{2y-1}\theta\,d\theta,$$

(d) $\displaystyle B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$

14.5 Triple Integrals

Now that we have seen how to extend definite integration to two-dimensional domains, the extension to three (or more) dimensions is straightforward. For a bounded function $f(x, y, z)$ defined on a rectangular box B ($x_0 \le x \le x_1$, $y_0 \le y \le y_1$, $z_0 \le z \le z_1$), the **triple integral** of f over B,

$$\iiint_B f(x, y, z)\, dV \qquad \text{or} \qquad \iiint_B f(x, y, z)\, dx\, dy\, dz,$$

can be defined as a suitable limit of Riemann sums corresponding to partitions of B into subboxes by planes parallel to each of the coordinate planes. We omit the details. Triple integrals over more general domains are defined by extending the function to be zero outside the domain and integrating over a rectangular box containing the domain.

All the properties of double integrals mentioned in Section 14.1 have analogues for triple integrals. In particular, a continuous function is integrable over a closed, bounded domain. If $f(x, y, z) = 1$ on the domain D, then the triple integral gives the volume of D:

$$\text{Volume of } D = \iiint_D dV.$$

The triple integral of a positive function $f(x, y, z)$ can be interpreted as the "hyper-volume" (i.e., the four-dimensional volume) of a region in 4-space having the set D as its three-dimensional "base" and having its top on the hypersurface $w = f(x, y, z)$. This is not a particularly useful interpretation; many more useful ones arise in applications. For instance, if $\delta(x, y, z)$ represents the density (mass per unit volume) at position (x, y, z) in a substance occupying the domain D in 3-space, then the mass m of the solid is the "sum" of mass elements $dm = \delta(x, y, z)\, dV$ occupying volume elements dV:

$$\text{mass} = \iiint_D \delta(x, y, z)\, dV.$$

Some triple integrals can be evaluated by inspection, using symmetry and known volumes.

Example 1 Evaluate

$$\iiint_{x^2+y^2+z^2 \le a^2} (2 + x - \sin z)\, dV.$$

Solution The domain of integration is the ball of radius a centred at the origin. The integral of 2 over this ball is twice the ball's volume, that is, $8\pi a^3/3$. The integrals of x and $\sin z$ over the ball are both zero, since both functions are odd in one of the variables and the domain is symmetric about each coordinate plane. (For instance, for every volume element dV in the half of the ball where $x > 0$, there is a corresponding element in the other half where x has the same size but the opposite sign. The contributions from these two elements cancel one another.) Thus,

$$\iiint_{x^2+y^2+z^2 \le a^2} (2 + x - \sin z)\, dV = \frac{8}{3}\pi a^3 + 0 + 0 = \frac{8}{3}\pi a^3.$$

■

Most triple integrals are evaluated by an iteration procedure similar to that used for double integrals. We slice the domain D with a plane parallel to one of the coordinate planes, double integrate the function with respect to two variables over that slice, and then integrate the result with respect to the remaining variable. Some examples should serve to make the procedure clear.

Solution As indicated in Figure 14.34(a), we will slice with planes perpendicular to the z-axis, so the z integral will be outermost in the iteration. The slices are rectangles, so the double integrals over them can be immediately iterated also. We do it with the y integral outer and the x integral inner, as suggested by the line shown in the slice.

$$\begin{aligned}
I &= \int_0^c dz \int_0^b dy \int_0^a (xy^2 + z^3)\,dx \\
&= \int_0^c dz \int_0^b dy \left(\frac{x^2y^2}{2} + xz^3\right)\Bigg|_{x=0}^{x=a} \\
&= \int_0^c dz \int_0^b \left(\frac{a^2y^2}{2} + az^3\right) dy \\
&= \int_0^c dz \left(\frac{a^2y^3}{6} + ayz^3\right)\Bigg|_{y=0}^{y=b} \\
&= \int_0^c \left(\frac{a^2b^3}{6} + abz^3\right) dz \\
&= \left(\frac{a^2b^3z}{6} + \frac{abz^4}{4}\right)\Bigg|_{z=0}^{z=c} = \frac{a^2b^3c}{6} + \frac{abc^4}{4}.
\end{aligned}$$

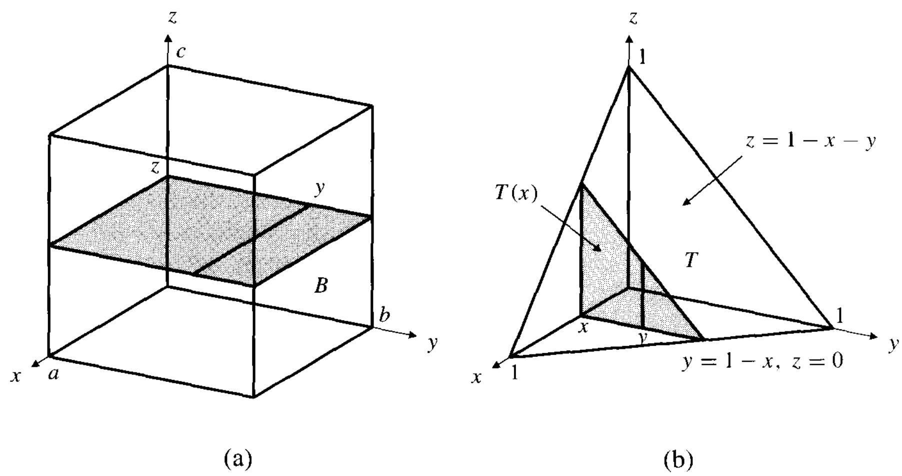


Figure 14.34

(a) The iteration in Example 2

(b) The iteration in Example 3

Example 3 If T is the tetrahedron with vertices $(0,0,0)$, $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$, evaluate

$$I = \iiint_T y\, dV.$$

Solution The tetrahedron is shown in Figure 14.34(b). The plane slice in the plane normal to the x-axis at position x is the triangle $T(x)$ shown in that figure; x is constant and y and z are variables in the slice. The double integral of y over $T(x)$ is a function of x. We evaluate it by integrating first in the z direction and then in the y direction as suggested by the vertical line shown in the slice:

$$\begin{aligned}\iint_{T(x)} y\, dA &= \int_0^{1-x} dy \int_0^{1-x-y} y\, dz \\ &= \int_0^{1-x} y(1-x-y)\, dy \\ &= \left((1-x)\frac{y^2}{2} - \frac{y^3}{3}\right)\Bigg|_0^{1-x} = \frac{1}{6}(1-x)^3.\end{aligned}$$

The value of the triple integral I is the integral of this expression with respect to the remaining variable x, to sum the contributions from all such slices between $x = 0$ and $x = 1$:

$$I = \int_0^1 \frac{1}{6}(1-x)^3\, dx = -\frac{1}{24}(1-x)^4\Bigg|_0^1 = \frac{1}{24}.$$

■

In the above solution we carried out the iteration in two steps in order to show the procedure clearly. In practice, triple integrals are iterated in one step, with no explicit mention made of the double integral over the slice. Thus, using the iteration suggested by Figure 14.34(b), we would immediately write

$$I = \int_0^1 dx \int_0^{1-x} dy \int_0^{1-x-y} y\, dz.$$

The evaluation proceeds as above, starting with the right (i.e., inner) integral, followed by the middle integral and then the left (outer) integral. The triple integral represents the "sum" of elements $y\, dV$ over the three-dimensional region T. The above iteration corresponds to "summing" (i.e., integrating) first along a vertical line (the z integral), then summing these one-dimensional sums in the y direction to get the double sum of all elements in the plane slice, and finally summing these double sums in the x direction to add up the contributions from all the slices. The iteration can be carried out in other directions; there are six possible iterations corresponding to different orders of doing the x, y, and z integrals. The other five are

$$I = \int_0^1 dx \int_0^{1-x} dz \int_0^{1-x-z} y\, dy,$$

$$I = \int_0^1 dy \int_0^{1-y} dx \int_0^{1-x-y} y\, dz,$$

$$I = \int_0^1 dy \int_0^{1-y} dz \int_0^{1-y-z} y\,dx,$$

$$I = \int_0^1 dz \int_0^{1-z} dx \int_0^{1-x-z} y\,dy,$$

$$I = \int_0^1 dz \int_0^{1-z} dy \int_0^{1-y-z} y\,dx.$$

You should verify these by drawing diagrams analogous to Figure 14.34(b). Of course, all six iterations give the same result.

It is sometimes difficult to visualize the region of 3-space over which a given triple integral is taken. In such situations try to determine the *projection* of that region on one or other of the coordinate planes. For instance, if a region R is bounded by two surfaces with given equations, combining these equations to eliminate one variable will yield the equation of a cylinder (not necessarily circular) with axis parallel to the axis of the eliminated variable. This cylinder will then determine the projection of R onto the coordinate plane perpendicular to that axis. The following example illustrates the use of this technique to find a volume bounded by two surfaces. The volume is expressed as a triple integral with unit integrand.

Example 4 Find the volume of the region R lying below the plane $z = 3 - 2y$ and above the paraboloid $z = x^2 + y^2$.

Solution The region R is shown in Figure 14.35. The two surfaces bounding R intersect on the vertical cylinder $x^2 + y^2 = 3 - 2y$, or $x^2 + (y+1)^2 = 4$. If D is the circular disk in which this cylinder intersects the xy-plane, then partial iteration gives

$$V = \iiint_R dV = \iint_D dx\,dy \int_{x^2+y^2}^{3-2y} dz.$$

Figure 14.35 shows a slice of R corresponding to a further iteration of the double integral over D:

$$V = \int_{-3}^{1} dy \int_{-\sqrt{3-2y-y^2}}^{\sqrt{3-2y-y^2}} dx \int_{x^2+y^2}^{3-2y} dz,$$

but there is an easier way to iterate the double integral. Since D is a circular disk of radius 2 and centre $(0, -1)$, we can use polar coordinates with centre at that point (i.e., $x = r\cos\theta$, $y = -1 + r\sin\theta$. Thus,

$$\begin{aligned} V &= \iint_D (3 - 2y - x^2 - y^2)\,dx\,dy \\ &= \iint_D \left(4 - x^2 - (y+1)^2\right) dx\,dy \\ &= \int_0^{2\pi} d\theta \int_0^2 (4 - r^2) r\,dr = 2\pi \left(2r^2 - \frac{r^4}{4}\right)\Bigg|_0^2 = 8\pi \text{ cubic units.} \end{aligned}$$

■

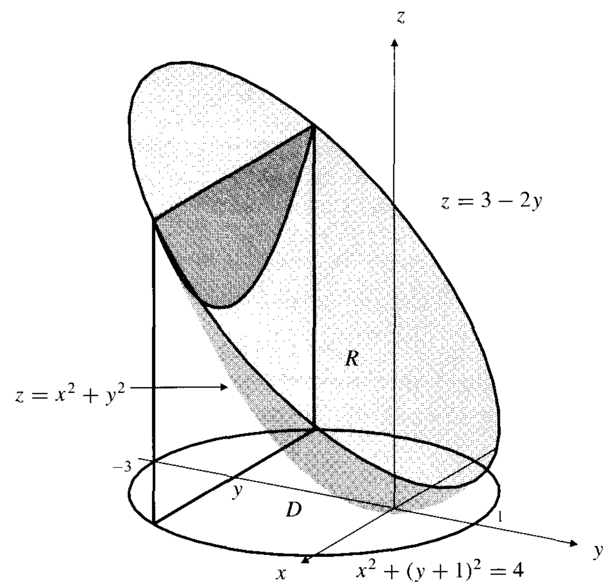


Figure 14.35 The volume above a paraboloid and under a slanting plane

As was the case for double integrals, it is sometimes necessary to reiterate a given iterated integral so that the integrations are performed in a different order. This task is most easily accomplished if we can translate the given iteration into a sketch of the region of integration. The ability to deduce the shape of the region from the limits in the iterated integral is a skill that one acquires with practice. You should first determine the projection of the region on a coordinate plane, namely, the plane of the two variables in the outer integrals of the given iteration.

It is also possible to reiterate an iterated integral in a different order by manipulating the limits of integration algebraically. We will illustrate both approaches (graphical and algebraic) in the following examples.

Example 5 Express the iterated integral

$$I = \int_0^1 dy \int_y^1 dz \int_0^z f(x, y, z)\, dx$$

as a triple integral, and sketch the region over which it is taken. Reiterate the integral in such a way that the integrations are performed in the order: first y, then z, then x (i.e., the opposite order to the given iteration).

Solution We express I as an uniterated triple integral:

$$I = \iiint_R f(x, y, z)\, dV.$$

The outer integral in the given iteration shows that the region R lies between the planes $y = 0$ and $y = 1$. For each such value of y, z must lie between y and 1. Therefore, R lies below the plane $z = 1$ and above the plane $z = y$, and the projection of R onto the yz-plane is the triangle with vertices $(0, 0, 0)$, $(0, 0, 1)$, and $(0, 1, 1)$. Through any point $(0, y, z)$ in this triangle, a line parallel to the x-axis intersects R between $x = 0$ and $x = z$. Thus, the solid is bounded by the five planes $x = 0$, $y = 0$, $z = 1$, $y = z$, and $z = x$. It is sketched in Figure 14.36(a), with slice and line corresponding to the given iteration.

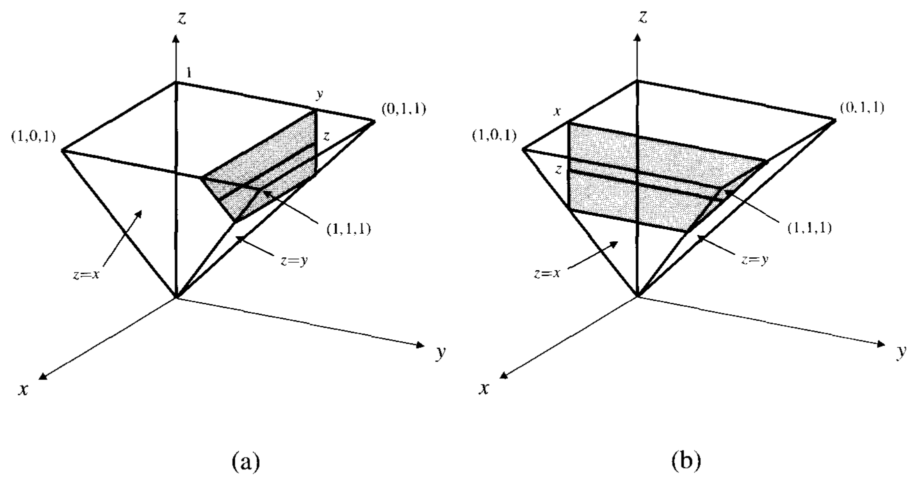


Figure 14.36
(a) The solid region for the triple integral in Example 5 sliced corresponding to the given iteration
(b) The same solid sliced to conform to the desired iteration

The required iteration corresponds to the slice and line shown in Figure 14.36(b). Therefore, it is

$$I = \int_0^1 dx \int_x^1 dz \int_0^z f(x, y, z)\, dy.$$

Example 6 Use algebra to write an iteration of the integral

$$I = \int_0^1 dx \int_x^1 dy \int_x^y f(x, y, z)\, dz$$

with the order of integrations reversed.

Solution From the given iteration we can write three sets of inequalities satisfied by the outer variable x, the middle variable y, and the inner variable z. We write these in order as follows:

$0 \le x \le 1$	inequalities for x
$x \le y \le 1$	inequalities for y
$x \le z \le y$	inequalities for z.

Note that the limits for each variable can be constant or can depend only on variables whose inequalities are on lines above the line for that variable. (In this case, the limits for x must both be constant, those for y can depend on x, and those for z can depend on both x and y.) This is a requirement for iterated integrals; outer integrals cannot depend on the variables of integration of the inner integrals.

We want to construct an equivalent set of inequalities with those for z on the top line, then those for y, then those for x on the bottom line. The limits for z must be constants. From the inequalities above we determine that $0 \le x \le z$ and $z \le y \le 1$. Thus z must satisfy $0 \le z \le 1$. The inequalities for y can depend on z. Since $z \le y$ and $y \le 1$, we have $z \le y \le 1$. Finally, the limits for x can depend on both y and z. We have $0 \le x$, $x \le y$, and $x \le z$. Since we have already determined that $z \le y$, we must have $0 \le x \le z$. Thus, the revised inequalities are

$$\begin{aligned} 0 \le z \le 1 & \quad \text{inequalities for } z \\ z \le y \le 1 & \quad \text{inequalities for } y \\ 0 \le x \le z & \quad \text{inequalities for } x \end{aligned}$$

and the required iteration is

$$I = \int_0^1 dz \int_z^1 dy \int_0^z f(x, y, z)\, dx.$$

Exercises 14.5

In Exercises 1–12, evaluate the triple integrals over the indicated region R. Be alert for simplifications and auspicious orders of iteration.

1. $\iiint_R (1 + 2x - 3y)\, dV$, over the box $-a \le x \le a$, $-b \le y \le b$, $-c \le z \le c$
2. $\iiint_B xyz\, dV$, over the box $0 \le x \le 1$, $-2 \le y \le 0$, $1 \le z \le 4$
3. $\iiint_D (3 + 2xy)\, dV$, over the solid hemispherical dome D given by $x^2 + y^2 + z^2 \le 4$ and $z \ge 0$
4. $\iiint_R x\, dV$, over the tetrahedron bounded by the coordinate planes and the plane $\dfrac{x}{a} + \dfrac{y}{b} + \dfrac{z}{c} = 1$
5. $\iiint_R (x^2 + y^2)\, dV$, over the cube $0 \le x, y, z \le 1$
6. $\iiint_R (x^2 + y^2 + z^2)\, dV$, over the cube of Exercise 5
7. $\iiint_R (xy + z^2)\, dV$, over the set $0 \le z \le 1 - |x| - |y|$
8. $\iiint_R yz^2 e^{-xyz}\, dV$, over the cube $0 \le x, y, z \le 1$
9. $\iiint_R \sin(\pi y^3)\, dV$, over the pyramid with vertices $(0, 0, 0)$, $(0, 1, 0)$, $(1, 1, 0)$, $(1, 1, 1)$, and $(0, 1, 1)$
10. $\iiint_R y\, dV$, over that part of the cube $0 \le x, y, z \le 1$ lying above the plane $y + z = 1$ and below the plane $x + y + z = 2$
11. $\iiint_R \dfrac{1}{(x + y + z)^3}\, dV$, over the region bounded by the six planes $z = 1$, $z = 2$, $y = 0$, $y = z$, $x = 0$, and $x = y + z$
12. $\iiint_R \cos x \cos y \cos z\, dV$, over the tetrahedron defined by $x \ge 0$, $y \ge 0$, $z \ge 0$, and $x + y + z \le \pi$
13. Evaluate $\iiint_{\mathbb{R}^3} e^{-x^2-2y^2-3z^2}\, dV$. *Hint:* use the result of Example 4 of Section 14.4.
14. Find the volume of the region lying inside the cylinder $x^2 + 4y^2 = 4$, above the xy-plane and below the plane $z = 2 + x$.
15. Find $\iiint_T x\, dV$, where T is the tetrahedron bounded by the planes $x = 1$, $y = 1$, $z = 1$ and $x + y + z = 2$.
16. Sketch the region R in the first octant of 3-space that has finite volume and is bounded by the surfaces $x = 0$, $z = 0$, $x + y = 1$, and $z = y^2$. Write six different iterations of the triple integral of $f(x, y, z)$ over R.

In Exercises 17–20, express the given iterated integral as a triple integral and sketch the region over which it is taken. Reiterate the integral so that the outermost integral is with respect to x and the innermost is with respect to z.

* 17. $\displaystyle\int_0^1 dz \int_0^{1-z} dy \int_0^1 f(x, y, z)\, dx$

* 18. $\displaystyle\int_0^1 dz \int_z^1 dy \int_0^y f(x, y, z)\, dx$

* 19. $\displaystyle\int_0^1 dz \int_z^1 dx \int_0^{x-z} f(x, y, z)\, dy$

* 20. $\displaystyle\int_0^1 dy \int_0^{\sqrt{1-y^2}} dz \int_{y^2+z^2}^1 f(x, y, z)\, dx$

21. Repeat Exercise 17 using the method of Example 6.
22. Repeat Exercise 18 using the method of Example 6.
23. Repeat Exercise 19 using the method of Example 6.
24. Repeat Exercise 20 using the method of Example 6.
25. Rework Example 5 using the method of Example 6.
26. Rework Example 6 using the method of Example 5.

In Exercises 27–28, evaluate the given iterated integral by reiterating it in a different order. (You will need to make a good sketch of the region.)

* **27.** $\displaystyle\int_0^1 dz \int_z^1 dx \int_0^x e^{x^3}\, dy$

* **28.** $\displaystyle\int_0^1 dx \int_0^{1-x} dy \int_y^1 \frac{\sin(\pi z)}{z(2-z)}\, dz$

29. Define the average value of an integrable function $f(x, y, z)$ over a region R of 3-space. Find the average value of $x^2 + y^2 + z^2$ over the cube $0 \le x \le 1$, $0 \le y \le 1$, $0 \le z \le 1$.

* **30.** State a Mean-Value Theorem for triple integrals analogous to Theorem 3 of Section 14.3. Use it to prove that if $f(x, y, z)$ is continuous near the point (a, b, c) and if $B_\epsilon(a, b, c)$ is the ball of radius ϵ centred at (a, b, c), then

$$\lim_{\epsilon \to 0} \frac{3}{4\pi\epsilon^3} \iiint_{B_\epsilon(a,b,c)} f(x, y, z)\, dV = f(a, b, c).$$

14.6 Change of Variables in Triple Integrals

The change of variables formula for a double integral extends to triple (and higher-order) integrals. Consider the transformation

$$\begin{aligned} x &= x(u, v, w), \\ y &= y(u, v, w), \\ z &= z(u, v, w), \end{aligned}$$

where x, y, and z have continuous first partial derivatives with respect to u, v, and w. Near any point where the Jacobian $\partial(x, y, z)/\partial(u, v, w)$ is nonzero, the transformation scales volume elements according to the formula

$$dV = dx\, dy\, dz = \left| \frac{\partial(x, y, z)}{\partial(u, v, w)} \right| du\, dv\, dw.$$

Thus, if the transformation is one-to-one from a domain S in uvw-space onto a domain D in xyz-space, and if

$$g(u, v, w) = f\big(x(u, v, w), y(u, v, w), z(u, v, w)\big),$$

then

$$\iiint_D f(x, y, z)\, dx\, dy\, dz = \iiint_S g(u, v, w) \left| \frac{\partial(x, y, z)}{\partial(u, v, w)} \right| du\, dv\, dw.$$

The proof is similar to that of the two-dimensional case given in Section 14.4. See Exercise 35 below.

Example 1 Under the change of variables $x = au$, $y = bv$, $z = cw$, where $a, b, c > 0$, the solid ellipsoid E given by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \le 1$$

becomes the ball B given by $u^2 + v^2 + w^2 \le 1$. Since the Jacobian of this transformation is

$$\frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} = abc,$$

the volume of the ellipsoid is given by

$$\begin{aligned}\text{Volume of } E &= \iiint_E dx\,dy\,dz \\ &= \iiint_B abc\,du\,dv\,dw = abc \times (\text{Volume of } B) \\ &= \frac{4}{3}\pi abc \text{ cubic units.}\end{aligned}$$

Cylindrical Coordinates

Among the most useful alternatives to Cartesian coordinates in 3-space are two coordinate systems that generalize plane polar coordinates. The simpler of these is the system of **cylindrical coordinates**. It uses ordinary plane polar coordinates in the xy-plane while retaining the Cartesian z coordinate for measuring vertical distances. Thus, each point in 3-space has cylindrical coordinates $[r, \theta, z]$ related to its Cartesian coordinates (x, y, z) by the transformation

$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad z = z.$$

Figure 14.37 shows how a point P is located by its cylindrical coordinates $[r, \theta, z]$ as well as by its Cartesian coordinates (x, y, z). Note that the distance from the origin to P is

$$d = \sqrt{r^2 + z^2} = \sqrt{x^2 + y^2 + z^2}.$$

Example 2 The point with Cartesian coordinates $(1, 1, 1)$ has cylindrical coordinates $[\sqrt{2}, \pi/4, 1]$. The point with Cartesian coordinates $(0, 2, -3)$ has cylindrical coordinates $[2, \pi/2, -3]$. The point with cylindrical coordinates $[4, -\pi/3, 5]$ has Cartesian coordinates $(2, -2\sqrt{3}, 5)$.

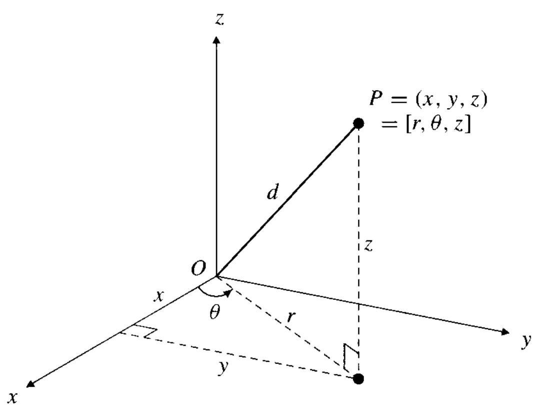


Figure 14.37 The cylindrical coordinates of a point

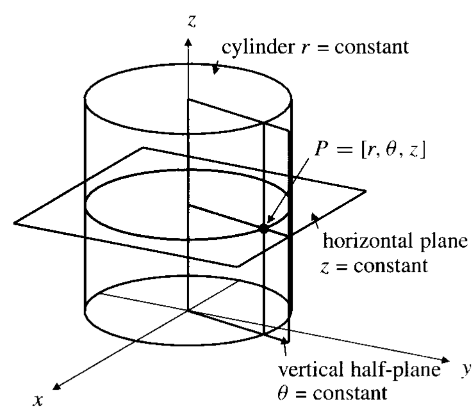


Figure 14.38 The coordinate surfaces for cylindrical coordinates

The coordinate surfaces in cylindrical coordinates are the r-surfaces (vertical circular cylinders centred on the z-axis), the θ-surfaces (vertical half-planes with edge along the z-axis), and the z-surfaces (horizontal planes). (See Figure 14.38.) Cylindrical coordinates lend themselves to representing domains that are bounded by such surfaces and, in general, to problems with axial symmetry (around the z-axis).

The **volume element in cylindrical coordinates** is

$$dV = r\,dr\,d\theta\,dz,$$

which is easily seen by examining the infinitesimal "box" bounded by the coordinate surfaces corresponding to values $r, r+dr, \theta, \theta+d\theta, z$, and $z+dz$ (see Figure 14.39), or by calculating the Jacobian

$$\frac{\partial(x,y,z)}{\partial(r,\theta,z)} = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r.$$

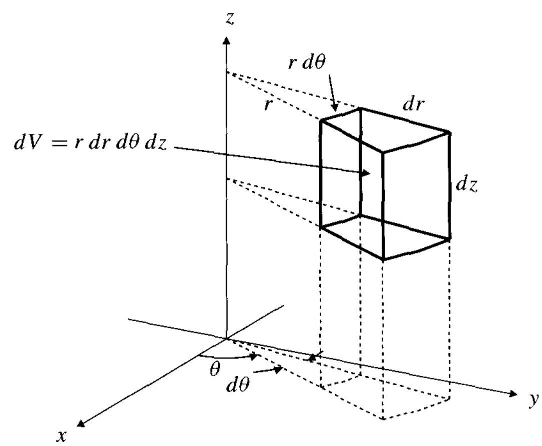


Figure 14.39 The volume element in cylindrical coordinates

Example 3 Evaluate

$$\iiint_D (x^2+y^2)\,dV$$

over the first octant region bounded by the cylinders $x^2+y^2=1$ and $x^2+y^2=4$ and the planes $z=0$, $z=1$, $x=0$, and $x=y$.

Solution In terms of cylindrical coordinates the region is bounded by $r=1$, $r=2$, $\theta=\pi/4$, $\theta=\pi/2$, $z=0$, and $z=1$. (See Figure 14.40. It is a rectangular coordinate box in $r\theta z$-space.) Since the integrand is $x^2+y^2=r^2$, the integral is

$$\begin{aligned}\iiint_D (x^2+y^2)\,dV &= \int_0^1 dz \int_{\pi/4}^{\pi/2} d\theta \int_1^2 r^2\,r\,dr \\ &= (1-0)\left(\frac{\pi}{2}-\frac{\pi}{4}\right)\left(\frac{2^4}{4}-\frac{1^4}{4}\right) = \frac{15}{16}\pi.\end{aligned}$$

This integral would have been much more difficult to evaluate using Cartesian coordinates.

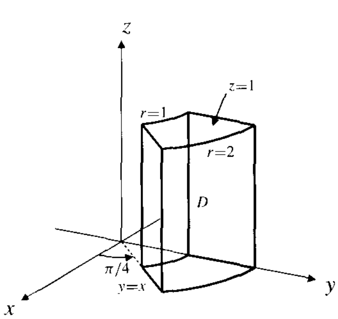


Figure 14.40

Example 4 Use a triple integral to find the volume of the solid region inside the sphere $x^2 + y^2 + z^2 = 6$ and above the paraboloid $z = x^2 + y^2$.

Solution One-quarter of the required volume lies in the first octant. (See region R in Figure 14.41.) The two surfaces intersect on the vertical cylinder

$$6 - x^2 - y^2 = z^2 = (x^2 + y^2)^2,$$

or, in terms of cylindrical coordinates,

$$\begin{aligned} &r^4 + r^2 - 6 = 0 \\ &(r^2 + 3)(r^2 - 2) = 0. \end{aligned}$$

The only relevant solution to this equation is $r = \sqrt{2}$. Thus, the required volume lies above the disk of radius $\sqrt{2}$ centred at the origin in the xy-plane. The total volume V is

$$\begin{aligned} V = \iiint_R dV &= \int_0^{2\pi} d\theta \int_0^{\sqrt{2}} r\,dr \int_{r^2}^{\sqrt{6-r^2}} dz \\ &= 2\pi \int_0^{\sqrt{2}} \left(r\sqrt{6-r^2} - r^3\right) dr \\ &= 2\pi \left[-\frac{1}{3}(6-r^2)^{3/2} - \frac{r^4}{4}\right]\Bigg|_0^{\sqrt{2}} \\ &= 2\pi \left[\frac{6\sqrt{6}}{3} - \frac{8}{3} - 1\right] = \frac{2\pi}{3}(6\sqrt{6} - 11) \text{ cubic units.} \end{aligned}$$

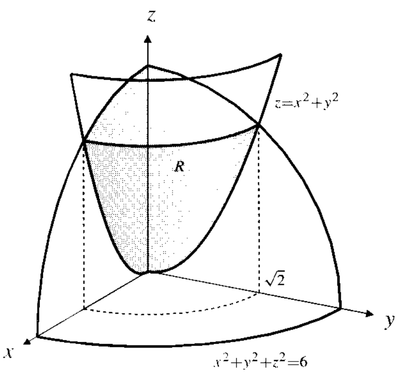


Figure 14.41

Spherical Coordinates

In the system of **spherical coordinates** a point P in 3-space is represented by the ordered triple $[\rho, \phi, \theta]$, where ρ is the distance from P to the origin O, ϕ is the angle the radial line OP makes with the positive direction of the z-axis, and θ is the angle between the plane containing P and the z-axis and the xz-plane. (See Figure 14.42.) It is conventional to consider spherical coordinates restricted in such a way that $\rho \ge 0$, $0 \le \phi \le \pi$, and $0 \le \theta < 2\pi$ (or $-\pi < \theta \le \pi$). Every point not on the z-axis then has exactly one spherical coordinate representation, and the transformation from Cartesian coordinates (x, y, z) to spherical coordinates $[\rho, \phi, \theta]$ is one-to-one off the z-axis. Using the right-angled triangles in the figure, we can see that this transformation is given by:

$$\begin{aligned} x &= \rho \sin\phi \cos\theta \\ y &= \rho \sin\phi \sin\theta \\ z &= \rho \cos\phi. \end{aligned}$$

Observe that

$$\rho^2 = x^2 + y^2 + z^2 = r^2 + z^2$$

and that the r coordinate in cylindrical coordinates is related to ρ and ϕ by

$$r = \sqrt{x^2 + y^2} = \rho \sin\phi.$$

Thus, also

$$\tan\phi = \frac{r}{z} = \frac{\sqrt{x^2 + y^2}}{z} \quad \text{and} \quad \tan\theta = \frac{y}{x}.$$

If $\phi = 0$ or $\phi = \pi$, then $r = 0$, so the θ coordinate is irrelevant at points on the z-axis.

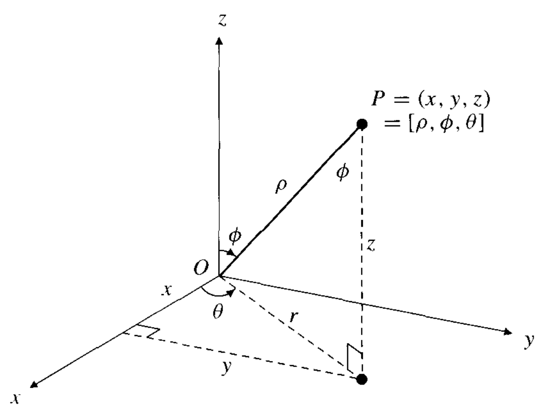


Figure 14.42 The spherical coordinates of a point

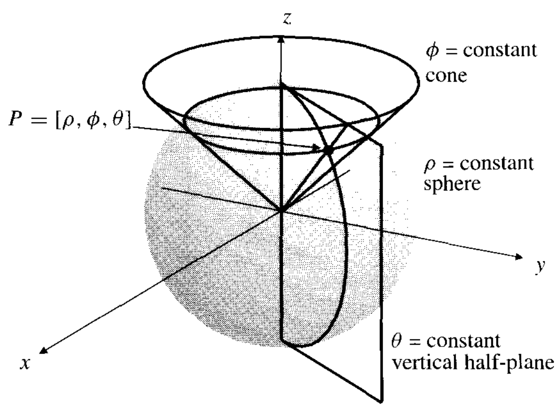


Figure 14.43 The coordinate surfaces for spherical coordinates

Some coordinate surfaces for spherical coordinates are shown in Figure 14.43. The ρ-surfaces (ρ = constant) are spheres centred at the origin; the ϕ-surfaces (ϕ = constant) are circular cones with the z-axis as axis; the θ-surfaces (θ = constant) are vertical half-planes with edge along the z-axis. If we take a coordinate system with origin at the centre of the earth, z-axis through the north pole, and x-axis through the intersection of the Greenwich meridian and the equator, then the intersections of the surface of the earth with the ϕ-surfaces are the *parallels of latitude*, and the intersections with the θ-surfaces are the *meridians of longitude*. Since latitude is measured from 90° at the north pole to −90° at the south pole, while ϕ is measured from 0 at the north pole to π (= 180°) at the south pole, the coordinate ϕ is frequently referred to as the **colatitude** coordinate. θ is the **longitude** coordinate. Observe that θ has the same significance in spherical coordinates as it does in cylindrical coordinates.

Example 5 Find:

(a) the Cartesian coordinates of the point P with spherical coordinates $[2, \pi/3, \pi/2]$ and

(b) the spherical coordinates of the point Q with Cartesian coordinates $(1, 1, \sqrt{2})$.

Solution

(a) If $\rho = 2$, $\phi = \pi/3$, and $\theta = \pi/2$, then

$$\begin{aligned} x &= 2\sin(\pi/3)\cos(\pi/2) = 0 \\ y &= 2\sin(\pi/3)\sin(\pi/2) = \sqrt{3} \\ z &= 2\cos(\pi/3) = 1. \end{aligned}$$

The Cartesian coordinates of P are $(0, \sqrt{3}, 1)$.

(b) Given that

$$\begin{aligned} \rho \sin\phi \cos\theta &= x = 1 \\ \rho \sin\phi \sin\theta &= y = 1 \\ \rho \cos\phi &= z = \sqrt{2}, \end{aligned}$$

we calculate that $\rho^2 = 1 + 1 + 2 = 4$, so $\rho = 2$. Also $r^2 = 1 + 1 = 2$, so $r = \sqrt{2}$. Thus $\tan\phi = r/z = 1$, so $\phi = \pi/4$. Also, $\tan\theta = y/x = 1$, so $\theta = \pi/4$ or $5\pi/4$. Since $x > 0$, we must have $\theta = \pi/4$. The spherical coordinates of Q are $[2, \pi/4, \pi/4]$.

Remark You may wonder why we write spherical coordinates in the order ρ, ϕ, θ rather than ρ, θ, ϕ. The reason, which will not become apparent until Chapter 16, is so that the triad of unit vectors at any point P pointing in the directions of increasing ρ, increasing ϕ, and increasing θ form a *right-handed basis* rather than a left-handed one.

The **volume element in spherical coordinates** is

$$dV = \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta.$$

To see this, observe that the infinitesimal coordinate box bounded by the coordinate surfaces corresponding to values ρ, $\rho + d\rho$, ϕ, $\phi + d\phi$, θ, and $\theta + d\theta$ has dimensions $d\rho$, $\rho \, d\phi$, and $\rho \sin\phi \, d\theta$. (See Figure 14.44.) Alternatively, the Jacobian of the transformation can be calculated:

$$\begin{aligned} \frac{\partial(x, y, z)}{\partial(\rho, \phi, \theta)} &= \begin{vmatrix} \sin\phi\cos\theta & \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\cos\phi\sin\theta & \rho\sin\phi\cos\theta \\ \cos\phi & -\rho\sin\phi & 0 \end{vmatrix} \\ &= \cos\phi \begin{vmatrix} \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \rho\cos\phi\sin\theta & \rho\sin\phi\cos\theta \end{vmatrix} \\ &\quad + \rho\sin\phi \begin{vmatrix} \sin\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\sin\phi\cos\theta \end{vmatrix} \\ &= \cos\phi(\rho^2\sin\phi\cos\phi) + \rho\sin\phi(\rho\sin^2\phi) \\ &= \rho^2\sin\phi. \end{aligned}$$

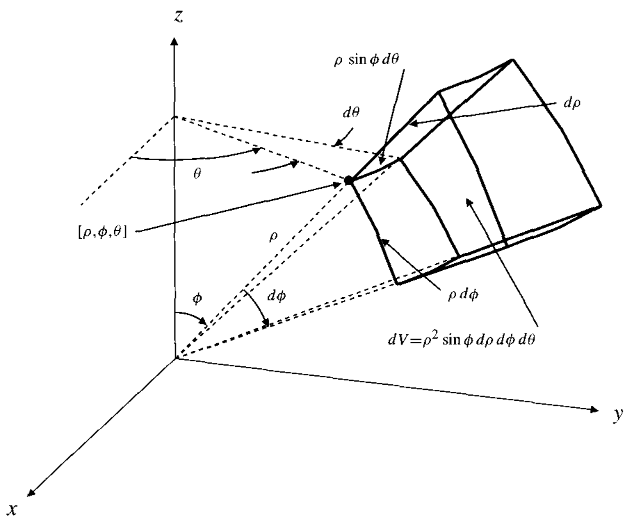


Figure 14.44 The volume element in spherical coordinates

Spherical coordinates are suited to problems involving spherical symmetry and, in particular, to regions bounded by spheres centred at the origin, circular cones with axes along the z-axis, and vertical planes containing the z-axis.

Example 6 A solid half-ball H of radius a has density depending on the distance ρ from the centre of the base disk. The density is given by $k(2a - \rho)$, where k is a constant. Find the mass of the half-ball.

Solution Choosing coordinates with origin at the centre of the base, and so that the half-ball lies above the xy-plane, we calculate the mass m as follows:

$$\begin{aligned} m &= \iiint_H k(2a-\rho)\, dV = \iiint_H k(2a-\rho)\, \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta \\ &= k\int_0^{2\pi} d\theta \int_0^{\pi/2} \sin\phi\, d\phi \int_0^a (2a-\rho)\, \rho^2\, d\rho \\ &= 2k\pi \times 1 \times \left(\frac{2a}{3}\rho^3 - \frac{1}{4}\rho^4\right)\bigg|_0^a = \frac{5}{6}\pi k a^4 \text{ units.} \end{aligned}$$

Remark In the above example both the integrand and the region of integration exhibited spherical symmetry, so the choice of spherical coordinates to carry out the integration was most appropriate. The mass could have been evaluated in cylindrical coordinates. The iteration in that system is

$$m = \int_0^{2\pi} d\theta \int_0^a r\, dr \int_0^{\sqrt{a^2-r^2}} k(2a - \sqrt{r^2+z^2})\, dz$$

and is much harder to evaluate. It is even more difficult in Cartesian coordinates:

$$m = 4\int_0^a dx \int_0^{\sqrt{a^2-x^2}} dy \int_0^{\sqrt{a^2-x^2-y^2}} k(2a - \sqrt{x^2+y^2+z^2})\, dz.$$

The choice of coordinate system can greatly affect the difficulty of computation of a multiple integral.

Many problems will have elements of spherical and axial symmetry. In such cases it may not be clear whether it would be better to use spherical or cylindrical coordinates. In such doubtful cases the integrand is usually the best guide. Use cylindrical or spherical coordinates according to whether the integrand involves $x^2 + y^2$ or $x^2 + y^2 + z^2$.

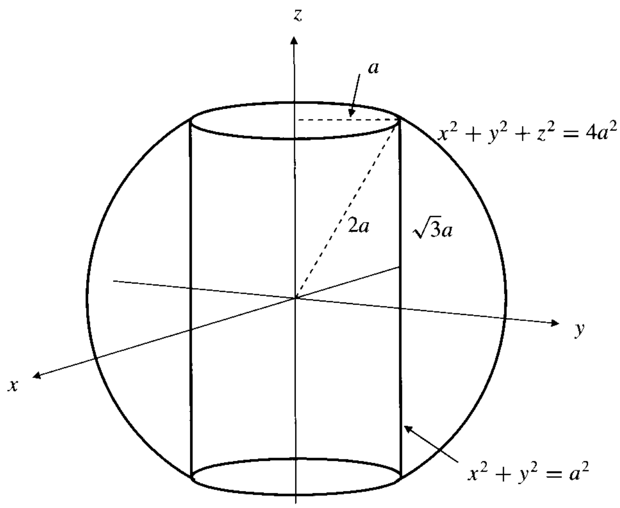


Figure 14.45 A solid ball with a cylindrical hole through it

Example 7 The moment of inertia about the z-axis of a solid of density δ occupying the region R is given by the integral

$$I = \iiint_R (x^2 + y^2)\delta \, dV.$$

(See Section 14.7.) Calculate that moment of inertia for a solid of unit density occupying the region inside the sphere $x^2 + y^2 + z^2 = 4a^2$ and outside the cylinder $x^2 + y^2 = a^2$.

Solution See Figure 14.45. In terms of spherical coordinates the required moment of inertia is

$$I = 2\int_0^{2\pi} d\theta \int_{\pi/6}^{\pi/2} \sin\phi \, d\phi \int_{a/\sin\phi}^{2a} \rho^2 \sin^2\phi \, \rho^2 \, d\rho.$$

In terms of cylindrical coordinates it is

$$I = 2\int_0^{2\pi} d\theta \int_a^{2a} r \, dr \int_0^{\sqrt{4a^2-r^2}} r^2 \, dz.$$

The latter formula looks somewhat easier to evaluate. We continue with it. Evaluating the θ and z integrals, we get

$$I = 4\pi \int_a^{2a} r^3 \sqrt{4a^2 - r^2} \, dr.$$

Making the substitution $u = 4a^2 - r^2$, $du = -2r \, dr$, we obtain

$$I = 2\pi \int_0^{3a^2} (4a^2 - u)\sqrt{u}\,du = 2\pi \left(4a^2 \frac{u^{3/2}}{3/2} - \frac{u^{5/2}}{5/2}\right)\bigg|_0^{3a^2} = \frac{44}{5}\sqrt{3}\pi a^5.$$

Exercises 14.6

1. Convert the spherical coordinates $[4, \pi/3, 2\pi/3]$ to Cartesian coordinates and to cylindrical coordinates.
2. Convert the Cartesian coordinates $(2, -2, 1)$ to cylindrical coordinates and to spherical coordinates.
3. Convert the cylindrical coordinates $[2, \pi/6, -2]$ to Cartesian coordinates and to spherical coordinates.
4. A point P has spherical coordinates $[1, \phi, \theta]$ and cylindrical coordinates $[r, \pi/4, r]$. Find the Cartesian coordinates of the point.

Describe the sets of points in 3-space that satisfy the equations in Exercises 5–14. Here, r, θ, ρ, and ϕ denote the appropriate cylindrical or spherical coordinates.

5. $\theta = \pi/2$
6. $\phi = 2\pi/3$
7. $\phi = \pi/2$
8. $\rho = 4$
9. $r = 4$
10. $\rho = z$
11. $\rho = r$
12. $\rho = 2x$
13. $\rho = 2\cos\phi$
14. $r = 2\cos\theta$

In Exercises 15–23, find the volumes of the indicated regions.

15. Inside the cone $z = \sqrt{x^2 + y^2}$ and inside the sphere $x^2 + y^2 + z^2 = a^2$
16. Above the surface $z = (x^2 + y^2)^{1/4}$ and inside the sphere $x^2 + y^2 + z^2 = 2$
17. Between the paraboloids $z = 10 - x^2 - y^2$ and $z = 2(x^2 + y^2 - 1)$
18. Inside the paraboloid $z = x^2 + y^2$ and inside the sphere $x^2 + y^2 + z^2 = 12$
19. Above the xy-plane, inside the cone $z = 2a - \sqrt{x^2 + y^2}$, and inside the cylinder $x^2 + y^2 = 2ay$
20. Above the xy-plane, under the paraboloid $z = 1 - x^2 - y^2$, and in the wedge $-x \le y \le \sqrt{3}x$
21. In the first octant, between the planes $y = 0$ and $y = x$, and inside the ellipsoid $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$. *Hint:* use the change of variables suggested in Example 1.

* 22. Bounded by the hyperboloid $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = 1$ and the planes $z = -c$ and $z = c$

23. Above the xy-plane and below the paraboloid

$$z = 1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}$$

24. Evaluate $\displaystyle\iiint_R (x^2 + y^2 + z^2)\,dV$, where R is the cylinder $0 \le x^2 + y^2 \le a^2, 0 \le z \le h$.
25. Find $\displaystyle\iiint_B (x^2 + y^2)\,dV$, where B is the ball given by $x^2 + y^2 + z^2 \le a^2$.
26. Find $\displaystyle\iiint_B (x^2 + y^2 + z^2)\,dV$, where B is the ball of Exercise 25.
27. Find $\displaystyle\iiint_R (x^2 + y^2 + z^2)\,dV$, where R is the region that lies above the cone $z = c\sqrt{x^2 + y^2}$ and inside the sphere $x^2 + y^2 + z^2 = a^2$.
28. Evaluate $\displaystyle\iiint_R (x^2 + y^2)\,dV$ over the region R of Exercise 27.
29. Find $\displaystyle\iiint_R z\,dV$, over the region R satisfying $x^2 + y^2 \le z \le \sqrt{2 - x^2 - y^2}$.
30. Find $\displaystyle\iiint_R x\,dV$ and $\displaystyle\iiint_R z\,dV$, over that part of the hemisphere $0 \le z \le \sqrt{a^2 - x^2 - y^2}$ that lies in the first octant.

* 31. Find $\displaystyle\iiint_R x\,dV$, and $\displaystyle\iiint_R z\,dV$ over that part of the cone

$$0 \le z \le h\left(1 - \frac{\sqrt{x^2 + y^2}}{a}\right)$$

that lies in the first octant.

* 32. Find the volume of the region inside the ellipsoid $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$ and above the plane $z = b - y$.

* 33. Show that for cylindrical coordinates the Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

is given by

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2} = 0.$$

* **34.** Show that in spherical coordinates the Laplace equation is given by

$$\frac{\partial^2 u}{\partial \rho^2}+\frac{2}{\rho}\frac{\partial u}{\partial \rho}+\frac{\cot\phi}{\rho^2}\frac{\partial u}{\partial \phi}+\frac{1}{\rho^2}\frac{\partial^2 u}{\partial \phi^2}+\frac{1}{\rho^2\sin^2\phi}\frac{\partial^2 u}{\partial \theta^2}=0.$$

* **35.** If x, y, and z are functions of u, v, and w with continuous first partial derivatives and nonvanishing Jacobian at (u, v, w), show that they map an infinitesimal volume element in uvw-space bounded by the coordinate planes u, $u + du$, v, $v + dv$, w, and $w + dw$ into an infinitesimal "parallelepiped" in xyz-space having volume

$$dx\,dy\,dz = \left|\frac{\partial(x, y, z)}{\partial(u, v, w)}\right| du\,dv\,dw.$$

Hint: adapt the two-dimensional argument given in Section 14.4. What three vectors from the point $P = (x(u, v, w), y(u, v, w), z(u, v, w))$ span the parallelepiped?

14.7 Applications of Multiple Integrals

When we express the volume V of a region R in 3-space as an integral,

$$V = \iiint_R dV,$$

we are regarding V as a "sum" of infinitely many *infinitesimal elements of volume,* that is, as the limit of the sum of volumes of smaller and smaller nonoverlapping subregions into which we subdivide R. This idea of representing sums of infinitesimal elements of quantities by integrals has many applications.

For example, if a rigid body of constant density δ g/cm^3 occupies a volume V cm^3, then its mass is $m = \delta V$ g. If the density is not constant but varies continuously over the region R of 3-space occupied by the rigid body, say $\delta = \delta(x, y, z)$, we can still regard the density as being constant on an infinitesimal element of R having volume dV. The mass of this element is therefore $dm = \delta(x, y, z)\,dV$, and the mass of the whole body is calculated by integrating these mass elements over R:

$$m = \iiint_R \delta(x, y, z)\,dV.$$

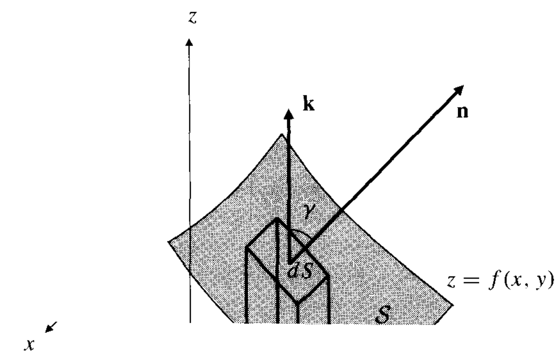
Similar formulas apply when the rigid body is one- or two-dimensional, and its density is given in units of mass per unit length or per unit area. In such cases single or double integrals are needed to sum the individual elements of mass. All this works because mass is "additive," that is, the mass of a composite object is the sum of the masses of the parts that comprise the object. The surface areas, gravitational forces, moments, and energies we consider in this section all have this additivity property.

The Surface Area of a Graph

We can use a double integral over a domain D in the xy-plane to add up surface area elements and thereby calculate the total area of the surface $\mathcal{S}$ with equation $z = f(x, y)$ defined for (x, y) in D. We assume that f has continuous first partial derivatives in D, so that $\mathcal{S}$ is smooth and has a nonvertical tangent plane at $P = \big(x, y, f(x, y)\big)$ for any (x, y) in D. The vector

$$\mathbf{n} = -f_1(x, y)\mathbf{i} - f_2(x, y)\mathbf{j} + \mathbf{k}$$

is an upward normal to $\mathcal{S}$ at P. An area element dA at position (x, y) in the xy-plane has a *vertical projection* onto $\mathcal{S}$ whose area dS is $\sec\gamma$ times the area dA, where γ is the angle between $\mathbf{n}$ and $\mathbf{k}$. (See Figure 14.46.)



is sec γ times as large as its projection dA onto the xy-plane

Since

$$\cos\gamma = \frac{\mathbf{n}\bullet\mathbf{k}}{|\mathbf{n}||\mathbf{k}|} = \frac{1}{\sqrt{1+\big(f_1(x,y)\big)^2+\big(f_2(x,y)\big)^2}},$$

we have

$$dS = \sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dA.$$

Therefore, the area of $\mathcal{S}$ is

$$S = \iint_D \sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dA.$$

Example 1 Find the area of that part of the hyperbolic paraboloid $z = x^2 - y^2$ that lies inside the cylinder $x^2 + y^2 = a^2$.

Solution Since $\partial z/\partial x = 2x$ and $\partial z/\partial y = -2y$, the surface area element is

$$dS = \sqrt{1+4x^2+4y^2}\,dA = \sqrt{1+4r^2}\,r\,dr\,d\theta.$$

The required surface area is the integral of dS over the disk $r \le a$:

$$\begin{aligned} S &= \int_0^{2\pi} d\theta \int_0^a \sqrt{1+4r^2}\,r\,dr \qquad\qquad (\text{Let } u = 1+4r^2.) \\ &= (2\pi)\frac{1}{8}\int_1^{1+4a^2} \sqrt{u}\,du \\ &= \frac{\pi}{4}\left(\frac{2}{3}\right)u^{3/2}\Big|_1^{1+4a^2} = \frac{\pi}{6}\left((1+4a^2)^{3/2}-1\right) \text{ square units.} \end{aligned}$$

■

The Gravitational Attraction of a Disk

Newton's universal law of gravitation asserts that two point masses m_1 and m_2, separated by a distance s, attract one another with a force

$$F = \frac{km_1m_2}{s^2},$$

k being a universal constant. The force on each mass is directed toward the other, along the line joining the two masses. Suppose that a flat disk D of radius a, occupying the region $x^2 + y^2 \le a^2$ of the xy-plane, has constant *areal density* σ (units of mass per unit area). Let us calculate the total force of attraction that this disk exerts upon a mass m located at the point $(0, 0, b)$ on the positive z-axis. The total force is a vector quantity. Although the various mass elements on the disk are in different directions from the mass m, symmetry indicates that the net force will be in the direction toward the centre of the disk, that is, toward the origin. Thus, the total force will be $-F\mathbf{k}$, where F is the magnitude of the force.

We will calculate F by integrating the vertical component dF of the force of attraction on m due to the mass $\sigma\, dA$ in an area element dA on the disk. If the area element is at the point with polar coordinates $[r, \theta]$, and if the line from this point to $(0, 0, b)$ makes angle ψ with the z-axis as shown in Figure 14.47, then the vertical component of the force of attraction of the mass element $\sigma\, dA$ on m is

$$dF = \frac{km\sigma\, dA}{r^2 + b^2} \cos\psi = km\sigma b \frac{dA}{(r^2 + b^2)^{3/2}}.$$

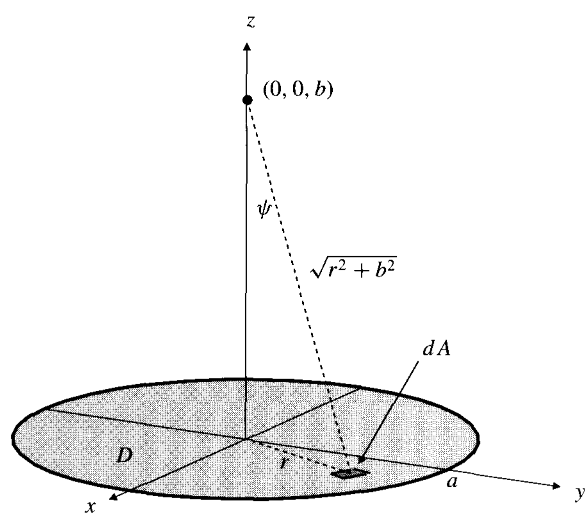


Figure 14.47 Each mass element $\sigma\, dA$ attracts m along a different line

Accordingly, the total vertical force of attraction of the disk on m is

$$\begin{aligned}
F &= km\sigma b \iint_D \frac{dA}{(r^2 + b^2)^{3/2}} \\
&= km\sigma b \int_0^{2\pi} d\theta \int_0^a \frac{r\, dr}{(r^2 + b^2)^{3/2}} \qquad \text{(Let } u = r^2 + b^2.\text{).} \\
&= \pi km\sigma b \int_{b^2}^{a^2+b^2} u^{-3/2}\, du \\
&= \pi km\sigma b \left(\frac{-2}{\sqrt{u}}\right)\Bigg|_{b^2}^{a^2+b^2} = 2\pi km\sigma \left(1 - \frac{b}{\sqrt{a^2 + b^2}}\right).
\end{aligned}$$

Remark If we let a approach infinity in the above formula, we obtain the formula $F = 2\pi k m \sigma$ for the force of attraction of a plane of areal density σ on a mass m located at distance b from the plane. Observe that F does not depend on b. Try to reason on physical grounds why this should be so.

Remark The force of attraction on a point mass due to suitably symmetric solid objects (such as balls, cylinders, and cones) having constant density δ (units of mass per unit volume) can be found by integrating elements of force contributed by thin, disk-shaped slices of the solid. See Exercises 14–17 below.

Moments and Centres of Mass

The centre of mass of a rigid body is that point (fixed in the body) at which the body can be supported so that in the presence of a constant gravitational field it will not experience any unbalanced torques that will cause it to rotate. The torques experienced by a mass element dm in the body can be expressed in terms of the **moments** of dm about the three coordinate planes. If the body occupies a region R in 3-space and has continuous volume density $\delta(x, y, z)$, then the mass element $dm = \delta(x, y, z)\, dV$ that occupies the volume element dV is said to have **moments** $(x - x_0)\, dm$, $(y - y_0)\, dm$, and $(z - z_0)\, dm$ about the planes $x = x_0$, $y = y_0$, and $z = z_0$, respectively. Thus, the total moments of the body about these three planes are

$$
\begin{aligned}
M_{x=x_0} &= \iiint_R (x - x_0)\delta(x, y, z)\, dV = M_{x=0} - x_0 m \\
M_{y=y_0} &= \iiint_R (y - y_0)\delta(x, y, z)\, dV = M_{y=0} - y_0 m \\
M_{z=z_0} &= \iiint_R (z - z_0)\delta(x, y, z)\, dV = M_{z=0} - z_0 m,
\end{aligned}
$$

where $m = \iiint_R \delta\, dV$ is the mass of the body and $M_{x=0}$, $M_{y=0}$, and $M_{z=0}$ are the moments about the coordinate planes $x = 0$, $y = 0$, and $z = 0$, respectively. The **centre of mass** $\bar{P} = (\bar{x}, \bar{y}, \bar{z})$ of the body is that point for which $M_{x=\bar{x}}$, $M_{y=\bar{y}}$, and $M_{z=\bar{z}}$ are all equal to zero. Thus,

Centre of mass

The centre of mass of a solid occupying region R of 3-space and having continuous density $\delta(x, y, z)$ (units of mass per unit volume) is the point $(\bar{x}, \bar{y}, \bar{z})$ with coordinates given by

$$
\bar{x} = \frac{M_{x=0}}{m} = \frac{\iiint_R x\delta\, dV}{\iiint_R \delta\, dV}, \qquad \bar{y} = \frac{M_{y=0}}{m} = \frac{\iiint_R y\delta\, dV}{\iiint_R \delta\, dV},
$$

$$
\bar{z} = \frac{M_{z=0}}{m} = \frac{\iiint_R z\delta\, dV}{\iiint_R \delta\, dV}.
$$

These formulas can be combined into a single vector formula for the position vector $\bar{\mathbf{r}} = \bar{x}\mathbf{i} + \bar{y}\mathbf{j} + \bar{z}\mathbf{k}$ of the centre of mass in terms of the position vector $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ of an arbitrary point in R,

$$\bar{\mathbf{r}} = \frac{M_{x=0}\mathbf{i} + M_{y=0}\mathbf{j} + M_{z=0}\mathbf{k}}{m} = \frac{\iiint_R \delta\,\mathbf{r}\,dV}{\iiint_R \delta\,dV},$$

where the integral of the vector function $\delta\,\mathbf{r}$ is understood to mean the vector whose components are the integrals of the components of $\delta\,\mathbf{r}$.

Remark Similar expressions hold for distributions of mass over regions in the plane or over intervals on a line. We use the appropriate areal or line densities and double or single definite integrals.

Remark If the density is constant, it cancels out of the expressions for the centre of mass. In this case the centre of mass is a *geometric* property of the region R and is called the **centroid** or **centre of gravity** of that region.

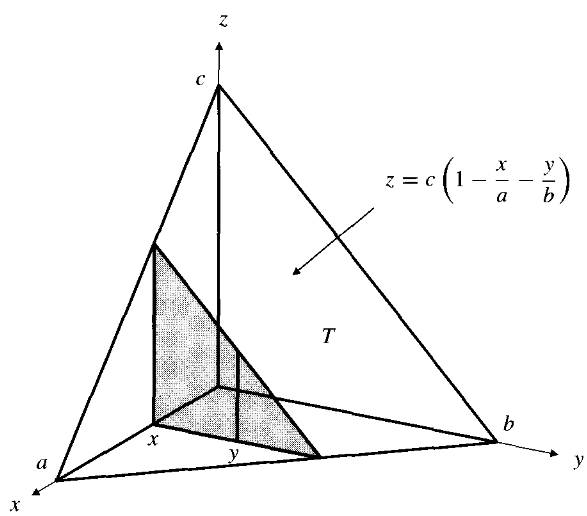


Figure 14.48 Iteration diagram for a triple integral over the tetrahedron of Example 2

Example 2 Find the centroid of the tetrahedron T bounded by the coordinate planes and the plane

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1.$$

Solution The density is assumed to be constant, so we may take it to be unity. The mass of T is thus equal to its volume: $m = V = abc/6$. The moment of T about the yz-plane is (see Figure 14.48):

$$
\begin{aligned}
M_{x=0} &= \iiint_T x\,dV \\
&= \int_0^a x\,dx \int_0^{b(1-\frac{x}{a})} dy \int_0^{c(1-\frac{x}{a}-\frac{y}{b})} dz \\
&= c\int_0^a x\,dx \int_0^{b(1-\frac{x}{a})} \left(1-\frac{x}{a}-\frac{y}{b}\right) dy \\
&= c\int_0^a x\left[\left(1-\frac{x}{a}\right)y - \frac{y^2}{2b}\right]\Bigg|_{y=0}^{y=b(1-\frac{x}{a})} dx \\
&= \frac{bc}{2}\int_0^a x\left(1-\frac{x}{a}\right)^2 dx \\
&= \frac{bc}{2}\left[\frac{x^2}{2} - \frac{2}{3}\frac{x^3}{a} + \frac{x^4}{4a^2}\right]\Bigg|_0^a = \frac{a^2bc}{24}.
\end{aligned}
$$

Thus $\bar{x} = M_{x=0}/m = a/4$. By symmetry, the centroid of T is $\left(\frac{a}{4}, \frac{b}{4}, \frac{c}{4}\right)$.

Example 3 Find the centre of mass of a solid occupying the region S that satisfies $x \ge 0$, $y \ge 0$, $z \ge 0$, and $x^2 + y^2 + z^2 \le a^2$, if the density at distance ρ from the origin is $k\rho$.

Solution The mass of the solid is distributed symmetrically in the first octant part of the ball $\rho \le a$ so that the centre of mass, $(\bar{x}, \bar{y}, \bar{z})$, must satisfy $\bar{x} = \bar{y} = \bar{z}$. The mass of the solid is

$$m = \iiint_S k\rho\,dV = \int_0^{\pi/2} d\theta \int_0^{\pi/2} \sin\phi\,d\phi \int_0^a (k\rho)\rho^2\,d\rho = \frac{\pi k a^4}{8}.$$

The moment about the xy-plane is

$$
\begin{aligned}
M_{z=0} &= \iiint_S zk\rho\,dV = \iiint_S (k\rho)\rho\cos\phi\,\rho^2\sin\phi\,d\rho\,d\phi\,d\theta \\
&= \frac{k}{2}\int_0^{\pi/2} d\theta \int_0^{\pi/2} \sin(2\phi)\,d\phi \int_0^a \rho^4\,d\rho = \frac{k\pi a^5}{20}.
\end{aligned}
$$

Hence, $\bar{z} = \dfrac{k\pi a^5}{20}\Big/\dfrac{k\pi a^4}{8} = \dfrac{2a}{5}$ and the centre of mass is $\left(\frac{2a}{5}, \frac{2a}{5}, \frac{2a}{5}\right)$.

Moment of Inertia

The **kinetic energy** of a particle of mass m moving with speed v is

$$\mathrm{KE} = \frac{1}{2}mv^2.$$

The mass of the particle measures its *inertia*, which is twice the energy it has when its speed is one unit.

If the particle is moving in a circle of radius D, its motion can be described in terms of its **angular speed**, Ω, measured in radians per unit time. In one revolution the particle travels a distance $2\pi D$ in time $2\pi/\Omega$. Thus, its (translational) speed v is related to its angular speed by

$$v = \Omega D.$$

Suppose that a rigid body is rotating with angular speed Ω about an axis L. If (at some instant) the body occupies a region R and has density $\delta = \delta(x, y, z)$, then each mass element $dm = \delta\, dV$ in the body has kinetic energy

$$d\mathrm{KE} = \frac{1}{2} v^2\, dm = \frac{1}{2}\, \delta \Omega^2 D^2\, dV,$$

where $D = D(x, y, z)$ is the perpendicular distance from the volume element dV to the axis of rotation L. The total kinetic energy of the rotating body is therefore

$$\mathrm{KE} = \frac{1}{2}\Omega^2 \iiint_R D^2 \delta\, dV = \frac{1}{2} I \Omega^2,$$

where

$$I = \iiint_R D^2 \delta\, dV.$$

I is called the **moment of inertia** of the rotating body about the axis L. The moment of inertia plays the same role in the expression for kinetic energy of rotation (in terms of angular speed) that the mass does in the expression for kinetic energy of translation (in terms of linear speed). The moment of inertia is twice the kinetic energy of the body when it is rotating with unit angular speed.

If the entire mass of the rotating body were concentrated at a distance D_0 from the axis of rotation, then its kinetic energy would be $\frac{1}{2} m D_0^2 \Omega^2$. The **radius of gyration** $\bar{D}$ is the value of D_0 for which this energy is equal to the actual kinetic energy $\frac{1}{2} I \Omega^2$ of the rotating body. Thus, $m\bar{D}^2 = I$ and the radius of gyration is

$$\bar{D} = \sqrt{I/m} = \left(\frac{\displaystyle\iiint_R D^2 \delta\, dV}{\displaystyle\iiint_R \delta\, dV} \right)^{1/2}.$$

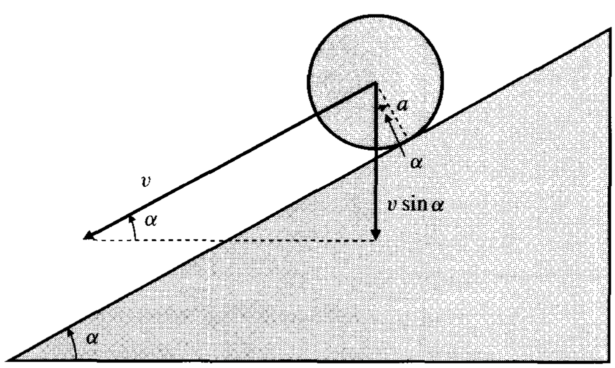


Figure 14.49 The actual velocity and the vertical velocity of a ball rolling down an incline as in Example 4

Example 4 **(The acceleration of a rolling ball)**

(a) Find the moment of inertia and radius of gyration of a solid ball of radius a and constant density δ about a diameter of that ball.

(b) With what linear acceleration will the ball roll (without slipping) down a plane inclined at angle α to the horizontal?

Solution

(a) We take the z-axis as the diameter and integrate in cylindrical coordinates over the ball B of radius a centred at the origin. Since the density δ is constant, we have

$$\begin{aligned} I &= \delta \iiint_B r^2\, dV \\ &= \delta \int_0^{2\pi} d\theta \int_0^a r^3\, dr \int_{-\sqrt{a^2-r^2}}^{\sqrt{a^2-r^2}} dz \\ &= 4\pi\delta \int_0^a r^3\sqrt{a^2-r^2}\, dr \qquad\qquad (\text{Let } u = a^2 - r^2.). \\ &= 2\pi\delta \int_0^{a^2} (a^2-u)\sqrt{u}\, du \\ &= 2\pi\delta \left(\frac{2}{3}a^2u^{3/2} - \frac{2}{5}u^{5/2}\right)\Bigg|_0^{a^2} = \frac{8}{15}\pi\delta a^5. \end{aligned}$$

Since the mass of the ball is $m = \frac{4}{3}\pi\delta a^3$, the radius of gyration is

$$\bar{D} = \sqrt{\frac{I}{m}} = \sqrt{\frac{2}{5}}\,a.$$

(b) We can determine the acceleration of the ball by using conservation of total (kinetic plus potential) energy. When the ball is rolling down the plane with speed v, its centre is moving with speed v and losing height at a rate $v\sin\alpha$. (See Figure 14.49.) Since the ball is not slipping, it is rotating about a horizontal axis through its centre with angular speed $\Omega = v/a$. Hence its kinetic energy (due to translation and rotation) is

$$\begin{aligned} \text{KE} &= \frac{1}{2}mv^2 + \frac{1}{2}I\Omega^2 \\ &= \frac{1}{2}mv^2 + \frac{1}{2}\frac{2}{5}ma^2\frac{v^2}{a^2} = \frac{7}{10}mv^2. \end{aligned}$$

When the centre of the ball is at height h (above some reference height) the ball has (gravitational) potential energy

$$\text{PE} = mgh.$$

(This is the work that must be done against a constant gravitational force $F = mg$ to raise it to height h.) Since total energy is conserved,

$$\frac{7}{10}mv^2 + mgh = \text{constant}.$$

Differentiating with respect to time t, we obtain

$$0 = \frac{7}{10}m\,2v\frac{dv}{dt} + mg\frac{dh}{dt} = \frac{7}{5}mv\,\frac{dv}{dt} - mgv\sin\alpha.$$

Thus, the ball rolls down the incline with acceleration $\dfrac{dv}{dt} = \dfrac{5}{7}g\sin\alpha$. ■

Exercises 14.7

Surface area problems

Use double integrals to calculate the areas of the surfaces in Exercises 1–9.

1. The part of the plane $z = 2x + 2y$ inside the cylinder $x^2 + y^2 = 1$

2. The part of the plane $5z = 3x - 4y$ inside the elliptic cylinder $x^2 + 4y^2 = 4$

3. The hemisphere $z = \sqrt{a^2 - x^2 - y^2}$

4. The half-ellipsoidal surface $z = 2\sqrt{1 - x^2 - y^2}$

5. The conical surface $3z^2 = x^2 + y^2$, $0 \le z \le 2$

6. The paraboloid $z = 1 - x^2 - y^2$ in the first octant

7. The part of the surface $z = y^2$ above the triangle with vertices $(0, 0)$, $(0, 1)$, and $(1, 1)$

8. The part of the surface $z = \sqrt{x}$ above the region $0 \le x \le 1$, $0 \le y \le \sqrt{x}$

9. The part of the cylindrical surface $x^2 + z^2 = 4$ that lies above the region $0 \le x \le 2$, $0 \le y \le x$

10. Show that the parts of the surfaces $z = 2xy$ and $z = x^2 + y^2$ that lie in the same vertical cylinder have the same area.

11. Show that the area S of the part of the paraboloid $z = \frac{1}{2}(x^2 + y^2)$ lying above the square $-1 \le x \le 1$, $-1 \le y \le 1$ is given by

$$S = \frac{8}{3}\int_0^{\pi/4} (1 + \sec^2\theta)^{3/2}\, d\theta - \frac{2\pi}{3},$$

and use numerical methods to evaluate the area to 3 decimal places.

* **12.** The canopy shown in Figure 14.50 is the part of the hemisphere of radius $\sqrt{2}$ centred at the origin that lies above the square $-1 \le x \le 1$, $-1 \le y \le 1$. Find its area. *Hint:* it is possible to get an exact solution by first finding the area of the part of the sphere $x^2 + y^2 + z^2 = 2$ that lies above the plane $z = 1$. If you do the problem directly by integrating the surface area element over the square, you may encounter an integral that you can't evaluate exactly, and you will have to use numerical methods.

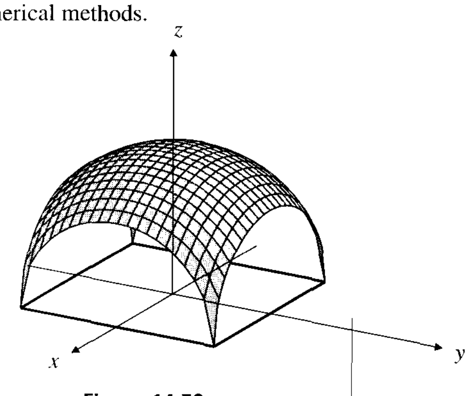


Figure 14.50

Mass and gravitational attraction

13. Find the mass of a spherical planet of radius a whose density at distance R from the centre is $\delta = A/(B + R^2)$.

In Exercises 14–17, find the gravitational attraction that the given object exerts on a mass m located at $(0, 0, b)$. Assume the object has constant density δ. In each case you can obtain the answer by integrating the contributions made by disks of thickness dz, making use of the formula for the attraction exerted by the disk obtained in the text.

14. The ball $x^2 + y^2 + z^2 \le a^2$, where $a < b$

15. The cylinder $x^2 + y^2 \le a^2$, $0 \le z \le h$, where $h < b$

16. The cone $0 \le z \le b - (\sqrt{x^2 + y^2})/a$

17. The half-ball $0 \le z \le \sqrt{a^2 - x^2 - y^2}$, where $a < b$

Centres of mass and centroids

18. Find the centre of mass of an object occupying the cube $0 \le x, y, z \le a$ with density given by $\delta = x^2 + y^2 + z^2$.

Find the centroids of the regions in Exercises 19–22.

19. The prism $x \ge 0$, $y \ge 0$, $x + y \le 1$, $0 \le z \le 1$

20. The unbounded region $0 \le z \le e^{-(x^2+y^2)}$

21. The first octant part of the ball $x^2 + y^2 + z^2 \le a^2$

22. The region inside the cube $0 \le x, y, z \le 1$ and under the plane $x + y + z = 2$

Moments of inertia

23. Explain *in physical terms* why the acceleration of the ball rolling down the incline in Example 4 does not approach g (the acceleration due to gravity) as the angle of incline, α, approaches $90°$.

Find the moments of inertia and radii of gyration of the solid objects in Exercises 24–32. Assume constant density in all cases.

24. A circular cylinder of base radius a and height h about the axis of the cylinder

25. A circular cylinder of base radius a and height h about a diameter of the base of the cylinder

26. A right circular cone of base radius a and height h about the axis of the cone

27. A right circular cone of base radius a and height h about a diameter of the base of the cone

28. A cube of edge length a about an edge of the cube

29. A cube of edge length a about a diagonal of a face of the cube

30. A cube of edge length a about a diagonal of the cube

31. The rectangular box $-a \le x \le a$, $-b \le y \le b$, $-c \le z \le c$ about the z-axis

32. The region between the two concentric cylinders $x^2 + y^2 = a^2$ and $x^2 + y^2 = b^2$ (where $0 < a < b$) and between $z = 0$ and $z = c$ about the z-axis

33. A ball of radius a has constant density δ. A cylindrical hole
a plane inclined at angle α to the horizontal. of the ball. Find

35. Repeat Exercise 34 for the ball with the cylindrical hole in Exercise 33. Assume that the axis of the hole remains horizontal while the ball rolls.

*** 36.** A rigid pendulum of mass m swings about point A on a horizontal axis. Its moment of inertia about that axis is I. The centre of mass C of the pendulum is at distance a from A. When the pendulum hangs at rest, C is directly under A. (Why?) Suppose the pendulum is swinging. Let $\theta = \theta(t)$ measure the angular displacement of the line AC from the vertical at time t. ($\theta = 0$ when the pendulum is in its rest position.) Use a conservation of energy argument similar to that in Example 4 to show that

$$\frac{1}{2} I \left(\frac{d\theta}{dt}\right)^2 - mga\cos\theta = \text{constant}$$

and, hence, differentiating with respect to t, that

$$\frac{d^2\theta}{dt^2} + \frac{mga}{I}\sin\theta = 0.$$

This is a nonlinear differential equation, and it is not easily solved. However, for small oscillations ($|\theta|$ small) we can use the approximation $\sin\theta \approx \theta$. In this case the differential equation is that of simple harmonic motion. What is the period?

*** 37.** Let L_0 be a straight line passing through the centre of mass of a rigid body B of mass m. Let L_k be a straight line passes through the point $(k, 0, 0)$.

*** 38.** Reestablish the expression for the total kinetic energy of the rolling ball in Example 4 by regarding the ball at any instant as rotating about a horizontal line through its point of contact with the inclined plane. Use the result of Exercise 37.

*** 39.** **(Products of inertia)** A rigid body with density δ is placed with its centre of mass at the origin and occupies a region R of 3-space. Suppose the six second moments P_{xx}, P_{yy}, P_{zz}, P_{xy}, P_{xz}, and P_{yz} are all known, where

$$P_{xx} = \iiint_R x^2\delta\, dV, \quad P_{xy} = \iiint_R xy\delta\, dV, \quad \ldots.$$

(There exist tables giving these six moments for bodies of many standard shapes. They are called products of inertia.) Show how to express the moment of inertia of the body about any axis through the origin in term of these six second moments. (If this result is combined with that of Exercise 37, the moment of inertia about *any* axis can be found.)

Chapter Review

Key Ideas

- **What do the following terms and phrases mean?**
  - ◇ a Riemann sum for $f(x, y)$ on $a \le x \le b$, $c \le y \le d$
  - ◇ $f(x, y)$ is integrable on $a \le x \le b$, $c \le y \le d$
  - ◇ the double integral of $f(x, y)$ over $a \le x \le b$, $c \le y \le d$
  - ◇ iteration of a double integral
  - ◇ the average value of $f(x, y)$ over region R
  - ◇ the area element in polar coordinates
  - ◇ a triple integral
  - ◇ the volume element in cylindrical coordinates
  - ◇ the volume element in spherical coordinates
  - ◇ the surface area of the graph of $z = f(x, y)$
  - ◇ the moment of inertia of a solid about an axis
- **Describe how to change variables in a double integral.**
- **How do you calculate the centroid of a solid region?**
- **How do you calculate the moment of inertia of a solid about an axis?**

Review Exercises

1. Evaluate $\iint_R (x+y)\,dA$, over the first-quadrant region lying under $x = y^2$ and above $y = x^2$.

2. Evaluate $\iint_P (x^2 + y^2)\,dA$, where P is the parallelogram with vertices $(0, 0)$, $(2, 0)$, $(3, 1)$, and $(1, 1)$.

3. Find $\iint_S (y/x)\,dA$, where S is the part of the disk $x^2+y^2 \le 4$ in the first quadrant and under the line $y = x$.

4. Consider the iterated integral

$$I = \int_0^{\sqrt{3}} dy \int_{y/\sqrt{3}}^{\sqrt{4-y^2}} e^{-x^2-y^2}\,dx.$$

(a) Write I as a double integral $\iint_R e^{-x^2-y^2}\,dA$, and sketch the region R over which the double integral is taken.

(b) Write I as an iterated integral with the order of integrations reversed from that of the given iteration.

(c) Write I as an iterated integral in polar coordinates.

(d) Evaluate I.

5. Find the constant $k > 0$ such that the volume of the region lying inside the sphere $x^2+y^2+z^2 = a^2$ and above the cone $z = k\sqrt{x^2+y^2}$ is one-quarter of the volume contained by the whole sphere.

6. Reiterate the integral

$$I = \int_0^2 dy \int_0^y f(x,y)\,dx + \int_2^6 dy \int_0^{\sqrt{6-y}} f(x,y)\,dx$$

with the y integral on the inside.

7. Let $J = \int_0^1 dz \int_0^z dy \int_0^y f(x,y,z)\,dx$. Express J as an iterated integral where the integrations are to be performed in the following order: first z, then y, then x.

8. An object in the shape of a right-circular cone has height 10 m and base radius 5 m. Its density is proportional to the square of the distance from the base and equals 3,000 kg/m^3 at the vertex.

(a) Find the mass of the object.

(b) Express the moment of inertia of the object about its central axis as an iterated integral.

9. Find the average value of $f(t) = \int_t^a e^{-x^2}\,dx$ over the interval $0 \le t \le a$.

10. Find the average value of the function $f(x,y) = \lfloor x+y \rfloor$ over the quarter-disk $x \ge 0$, $y \ge 0$, $x^2+y^2 \le 4$. (Recall that $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x.)

11. Let D be the smaller of the two solid regions bounded by the surfaces

$$z = \frac{x^2+y^2}{a} \quad \text{and} \quad x^2+y^2+z^2 = 6a^2,$$

where a is a positive constant. Find $\iiint_D (x^2+y^2)\,dV$.

12. Find the moment of inertia about the z-axis of a solid V of density 1 if V is specified by the inequalities $0 \le z \le \sqrt{x^2+y^2}$ and $x^2+y^2 \le 2ay$, where $a > 0$.

13. The rectangular solid $0 \le x \le 1$, $0 \le y \le 2$, $0 \le z \le 1$ is cut into two pieces by the plane $2x+y+z = 2$. Let D be the piece that includes the origin. Find the volume of D and $\bar{z}$, the z-coordinate of the centroid of D.

14. A solid S consists of those points (x,y,z) that lie in the first octant and satisfy $x+y+2z \le 2$ and $y+z \le 1$. Find the volume of S and the x-coordinate of its centroid.

15. Find $\iiint_S z\,dV$, where S is the portion of the first octant that is above the plane $x+y-z = 1$ and below the plane $z = 1$.

16. Find the area of that part of the plane $z = 2x$ that lies inside the paraboloid $z = x^2+y^2$.

17. Find the area of that part of the paraboloid $z = x^2+y^2$ that lies below the plane $z = 2x$. Express the answer as a single integral and evaluate it to 3 decimal places.

* 18. Find the volume of the smaller of the two regions into which the plane $x+y+z = 1$ divides the interior of the ellipsoid $x^2+4y^2+9z^2 = 36$. *Hint:* first change variables so that the ellipsoid becomes a ball. Then replace the plane by a plane with a simpler equation passing the same distance from the origin.

Challenging Problems

1. The plane $(x/a)+(y/b)+(z/c) = 1$ (where $a > 0$, $b > 0$, and $c > 0$) divides the solid ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} \le 1$$

into two unequal pieces. Find the volume of the smaller piece.

2. Find the area of the part of the plane $(x/a)+(y/b)+(z/c) = 1$ (where $a > 0$, $b > 0$, and $c > 0$) that lies inside the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} \le 1.$$

3. (a) Expand $1/(1-xy)$ as a geometric series, and hence show that

$$\int_0^1\int_0^1 \frac{1}{1-xy}\,dx\,dy = \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

(b) Similarly, express the following integrals as sums of series:

(i) $\displaystyle\int_0^1\int_0^1 \frac{1}{1+xy}\,dx\,dy,$

(ii) $\displaystyle\int_0^1\int_0^1\int_0^1 \frac{1}{1-xyz}\,dx\,dy\,dz,$

(iii) $\displaystyle\int_0^1\int_0^1\int_0^1 \frac{1}{1+xyz}\,dx\,dy\,dz.$

* 4. Let P be the parallelepiped bounded by the three pairs of parallel planes $\mathbf{a}\bullet\mathbf{r} = 0$, $\mathbf{a}\bullet\mathbf{r} = d_1 > 0$, $\mathbf{b}\bullet\mathbf{r} = 0$, $\mathbf{b}\bullet\mathbf{r} = d_2 > 0$, $\mathbf{c}\bullet\mathbf{r} = 0$, and $\mathbf{c}\bullet\mathbf{r} = d_3 > 0$, where $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are constant vectors, and $\mathbf{r} = x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$. Show that

$$\iiint_P (\mathbf{a}\bullet\mathbf{r})(\mathbf{b}\bullet\mathbf{r})(\mathbf{c}\bullet\mathbf{r})\,dx\,dy\,dz = \frac{(d_1d_2d_3)^2}{8|\mathbf{a}\bullet(\mathbf{b}\times\mathbf{c})|}.$$

Hint: make the change of variables $u = \mathbf{a} \bullet \mathbf{r}$, $v = \mathbf{b} \bullet \mathbf{r}$, $w = \mathbf{c} \bullet \mathbf{r}$.

5. A hole whose cross-section is a square of side 2 is punched through the middle of a ball of radius 2. Find the volume of the remaining part of the ball.

* 6. Find the volume bounded by the surface with equation $x^{2/3} + y^{2/3} + z^{2/3} = a^{2/3}$.

* 7. Find the volume bounded by the surface $|x|^{1/3} + |y|^{1/3} + |z|^{1/3} = |a|^{1/3}$.