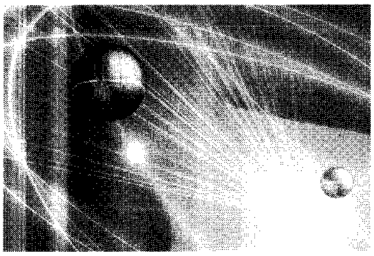

CHAPTER 16

Vector Calculus

Introduction In this chapter we develop two- and three-dimensional analogues of the one-dimensional Fundamental Theorem of Calculus. These analogues—Green's Theorem, Gauss's Divergence Theorem, and Stokes's Theorem—are of great importance both theoretically and in applications. They are phrased in terms of certain differential operators, divergence and curl, which are related to the gradient operator encountered in Section 12.7. The operators are introduced and their properties are derived in Sections 16.1 and 16.2. The rest of the chapter deals with the generalizations of the Fundamental Theorem of Calculus and their applications.

16.1 Gradient, Divergence, and Curl

First-order information about the rate of change of a 3-dimensional scalar field, $f(x, y, z)$, is contained in the three first partial derivatives $\partial f/\partial x$, $\partial f/\partial y$, and $\partial f/\partial z$. The gradient,

$$\mathbf{grad}\, f(x, y, z) = \nabla f(x, y, z) = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k},$$

collects this information into a single vector-valued "derivative" of f. We would like to develop similar ways of conveying information about the rate of change of vector fields.

First-order information about the rate of change of the vector field

$$\mathbf{F}(x, y, z) = F_1(x, y, z)\mathbf{i} + F_2(x, y, z)\mathbf{j} + F_3(x, y, z)\mathbf{k}$$

is contained in nine first partial derivatives, three for each of the three components of $\mathbf{F}$:

$$\begin{matrix} \dfrac{\partial F_1}{\partial x} & \dfrac{\partial F_1}{\partial y} & \dfrac{\partial F_1}{\partial z} \\[2ex] \dfrac{\partial F_2}{\partial x} & \dfrac{\partial F_2}{\partial y} & \dfrac{\partial F_2}{\partial z} \\[2ex] \dfrac{\partial F_3}{\partial x} & \dfrac{\partial F_3}{\partial y} & \dfrac{\partial F_3}{\partial z}. \end{matrix}$$

(Again, we stress that F_1, F_2, and F_3 denote the components of $\mathbf{F}$, not partial derivatives.) Two special combinations of these derivatives organize this information in particularly useful ways, as the gradient does for scalar fields. These are the **divergence** of $\mathbf{F}$ (**div** $\mathbf{F}$) and the **curl** of $\mathbf{F}$ (**curl** $\mathbf{F}$), defined as follows:

Divergence and curl

$$\operatorname{div}\mathbf{F} = \nabla \bullet \mathbf{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z},$$

$$\begin{aligned}\operatorname{curl}\mathbf{F} &= \nabla\times\mathbf{F}\\ &= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\mathbf{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\mathbf{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\mathbf{k}\\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}.\end{aligned}$$

Note that the divergence of a vector field is a scalar field, while the curl is another vector field. Also observe the notation $\nabla \bullet \mathbf{F}$ and $\nabla\times\mathbf{F}$, which we will sometimes use instead of **div F** and **curl F**. This makes use of the *vector differential operator*

$$\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z},$$

frequently called *del* or *nabla*. Just as the gradient of the scalar field f can be regarded as *formal scalar multiplication* of ∇ and f, so also can the divergence and curl of **F** be regarded as *formal dot* and *cross products* of ∇ with **F**. When using ∇ the order of "factors" is important; the quantities on which ∇ acts must appear to the right of ∇. For instance, $\nabla \bullet \mathbf{F}$ and $\mathbf{F} \bullet \nabla$ do not mean the same thing; the former is a scalar field and the latter is a scalar differential operator:

BEWARE Do not confuse the scalar field $\nabla \bullet \mathbf{F}$ with the scalar differential operator $\mathbf{F} \bullet \nabla$. They are quite different objects.

$$\mathbf{F} \bullet \nabla = F_1\frac{\partial}{\partial x} + F_2\frac{\partial}{\partial y} + F_3\frac{\partial}{\partial z}.$$

Example 1 Find the divergence and curl of the vector field

$$\mathbf{F} = xy\mathbf{i} + (y^2 - z^2)\mathbf{j} + yz\mathbf{k}.$$

Solution We have

$$\operatorname{div}\mathbf{F} = \nabla \bullet \mathbf{F} = \frac{\partial}{\partial x}(xy) + \frac{\partial}{\partial y}(y^2 - z^2) + \frac{\partial}{\partial z}(yz) = y + 2y + y = 4y,$$

$$\begin{aligned}\operatorname{curl}\mathbf{F} = \nabla\times\mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & y^2 - z^2 & yz \end{vmatrix}\\ &= \left[\frac{\partial}{\partial y}(yz) - \frac{\partial}{\partial z}(y^2 - z^2)\right]\mathbf{i} + \left[\frac{\partial}{\partial z}(xy) - \frac{\partial}{\partial x}(yz)\right]\mathbf{j}\\ &\quad + \left[\frac{\partial}{\partial x}(y^2 - z^2) - \frac{\partial}{\partial y}(xy)\right]\mathbf{k} = 3z\mathbf{i} - x\mathbf{k}.\end{aligned}$$

■

The divergence and curl of a two-dimensional vector field can also be defined: if $\mathbf{F}(x, y) = F_1(x, y)\mathbf{i} + F_2(x, y)\mathbf{j}$, then

$$\textbf{div}\,\mathbf{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y},$$
$$\textbf{curl}\,\mathbf{F} = \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\mathbf{k}.$$

Note that the curl of a two-dimensional vector field is still a 3-vector and is perpendicular to the plane of the field. Although **div** and **grad** are defined in all dimensions, **curl** is defined only in three dimensions and in the plane (provided we allow values in three dimensions).

Example 2 Find the divergence and curl of $\mathbf{F} = xe^y\mathbf{i} - ye^x\mathbf{j}$.

Solution We have

$$\textbf{div}\,\mathbf{F} = \nabla \bullet \mathbf{F} = \frac{\partial}{\partial x}(xe^y) + \frac{\partial}{\partial y}(-ye^x) = e^y - e^x,$$
$$\textbf{curl}\,\mathbf{F} = \nabla \times \mathbf{F} = \left(\frac{\partial}{\partial x}(-ye^x) - \frac{\partial}{\partial y}(xe^y)\right)\mathbf{k}$$
$$= -(ye^x + xe^y)\mathbf{k}.$$

■

Interpretation of the Divergence

The value of the divergence of a vector field $\mathbf{F}$ at point P is, loosely speaking, a measure of the rate at which the field "diverges" or "spreads away" from P. This spreading away can be measured by the flux out of a small closed surface surrounding P. For instance, $\textbf{div}\,\mathbf{F}(P)$ is the limit of the *flux per unit volume* out of smaller and smaller spheres centred at P.

THEOREM 1

The divergence as flux density

If $\hat{\mathbf{N}}$ is the unit outward normal on the sphere $\mathcal{S}_\epsilon$ of radius ϵ centred at point P, and if $\mathbf{F}$ is a smooth three-dimensional vector field, then

$$\textbf{div}\,\mathbf{F}(P) = \lim_{\epsilon \to 0^+} \frac{3}{4\pi\epsilon^3} \oiint_{\mathcal{S}_\epsilon} \mathbf{F} \bullet \hat{\mathbf{N}}\, dS.$$

PROOF Without loss of generality we assume that P is at the origin. We want to expand $\mathbf{F}$ in a Taylor series about the origin (a Maclaurin series). As shown in Section 12.9 for a function of two variables, the Maclaurin series for a scalar-valued function of three variables takes the form

$$f(x, y, z) = f(0,0,0) + \left.\frac{\partial f}{\partial x}\right|_{(0,0,0)} x + \left.\frac{\partial f}{\partial y}\right|_{(0,0,0)} y + \left.\frac{\partial f}{\partial z}\right|_{(0,0,0)} z + \cdots,$$

where "$\cdots$" represents terms of second and higher degree in x, y, and z. If we apply this formula to the components of $\mathbf{F}$, we obtain

$$\mathbf{F}(x, y, z) = \mathbf{F}_0 + \mathbf{F}_1 x + \mathbf{F}_2 y + \mathbf{F}_3 z + \cdots,$$

$$\mathbf{F}_2 = \frac{\partial \mathbf{F}}{\partial y}\bigg|_{(0,0,0)} = \left(\frac{\partial F_1}{\partial y}\mathbf{i} + \frac{\partial F_2}{\partial y}\mathbf{j} + \frac{\partial F_3}{\partial y}\mathbf{k}\right)\bigg|_{(0,0,0)}$$

$$\mathbf{F}_3 = \frac{\partial \mathbf{F}}{\partial z}\bigg|_{(0,0,0)} = \left(\frac{\partial F_1}{\partial z}\mathbf{i} + \frac{\partial F_2}{\partial z}\mathbf{j} + \frac{\partial F_3}{\partial z}\mathbf{k}\right)\bigg|_{(0,0,0)};$$

again, the "$\cdots$" represents the second- and higher-degree terms in x, y, and z. The unit normal on $\mathcal{S}_\epsilon$ is $\hat{\mathbf{N}} = (x\mathbf{i} + y\mathbf{j} + z\mathbf{k})/\epsilon$, so we have

$$\begin{aligned}\mathbf{F} \bullet \hat{\mathbf{N}} = \frac{1}{\epsilon}\Big(&\mathbf{F}_0 \bullet \mathbf{i}\,x + \mathbf{F}_0 \bullet \mathbf{j}\,y + \mathbf{F}_0 \bullet \mathbf{k}\,z \\ &+ \mathbf{F}_1 \bullet \mathbf{i}\,x^2 + \mathbf{F}_1 \bullet \mathbf{j}\,xy + \mathbf{F}_1 \bullet \mathbf{k}\,xz \\ &+ \mathbf{F}_2 \bullet \mathbf{i}\,xy + \mathbf{F}_2 \bullet \mathbf{j}\,y^2 + \mathbf{F}_2 \bullet \mathbf{k}\,yz \\ &+ \mathbf{F}_3 \bullet \mathbf{i}\,xz + \mathbf{F}_3 \bullet \mathbf{j}\,yz + \mathbf{F}_3 \bullet \mathbf{k}\,z^2 + \cdots\Big).\end{aligned}$$

We integrate each term within the parentheses over $\mathcal{S}_\epsilon$. By symmetry,

$$\begin{aligned}&\oiint_{\mathcal{S}_\epsilon} x\,dS = \oiint_{\mathcal{S}_\epsilon} y\,dS = \oiint_{\mathcal{S}_\epsilon} z\,dS = 0,\\ &\oiint_{\mathcal{S}_\epsilon} xy\,dS = \oiint_{\mathcal{S}_\epsilon} xz\,dS = \oiint_{\mathcal{S}_\epsilon} yz\,dS = 0.\end{aligned}$$

Also, by symmetry,

$$\begin{aligned}\oiint_{\mathcal{S}_\epsilon} x^2\,dS &= \oiint_{\mathcal{S}_\epsilon} y^2\,dS = \oiint_{\mathcal{S}_\epsilon} z^2\,dS \\ &= \frac{1}{3}\oiint_{\mathcal{S}_\epsilon} (x^2 + y^2 + z^2)\,dS = \frac{1}{3}(\epsilon^2)(4\pi\epsilon^2) = \frac{4}{3}\pi\epsilon^4,\end{aligned}$$

and the higher-degree terms have surface integrals involving ϵ^5 and higher powers. Thus,

$$\begin{aligned}\frac{3}{4\pi\epsilon^3}\oiint_{\mathcal{S}_\epsilon} \mathbf{F} \bullet \hat{\mathbf{N}}\,dS &= \mathbf{F}_1 \bullet \mathbf{i} + \mathbf{F}_2 \bullet \mathbf{j} + \mathbf{F}_3 \bullet \mathbf{k} + \epsilon\,(\cdots) \\ &= \nabla \bullet \mathbf{F}(0,0,0) + \epsilon\,(\cdots) \\ &\to \nabla \bullet \mathbf{F}(0,0,0)\end{aligned}$$

as $\epsilon \to 0^+$. This is what we wanted to show.

Remark The spheres $\mathcal{S}_\epsilon$ in the above theorem can be replaced by other contracting families of piecewise smooth surfaces. For instance, if B is the surface of a rectangular box with dimensions Δx, Δy, and Δz containing P, then

$$\mathbf{div}\,\mathbf{F}(P) = \lim_{\Delta x, \Delta y, \Delta z \to 0} \frac{1}{\Delta x \Delta y \Delta z} \oiint_B \mathbf{F} \bullet \hat{\mathbf{N}}\,dS.$$

See Exercise 12 below.

Remark In two dimensions, the value $\mathbf{div}\,\mathbf{F}(P)$ represents the limiting *flux per unit area* outward across small, non-self-intersecting closed curves which enclose P. See Exercise 13 at the end of this section.

Let us return again to the interpretation of a vector field as a velocity field of a moving incompressible fluid. If the total flux of the velocity field outward across the boundary surface of a domain is positive (or negative), then the fluid must be produced (or annihilated) within that domain.

The vector field $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ of Example 2 in Section 15.6 has constant divergence, $\nabla \bullet \mathbf{F} = 3$. In that example we showed that the flux of $\mathbf{F}$ out of a certain cylinder of base radius a and height $2h$ is $6\pi a^2 h$, which is three times the volume of the cylinder. Exercises 2 and 3 of Section 15.6 confirm similar results for the flux of $\mathbf{F}$ out of other domains. This leads to another interpretation for the divergence; $\mathbf{div}\,\mathbf{F}(P)$ is the *source strength per unit volume* of $\mathbf{F}$ at P. With this interpretation, we would expect, even for a vector field $\mathbf{F}$ with nonconstant divergence, that the total flux of $\mathbf{F}$ out of the surface $\mathcal{S}$ of a domain D would be equal to the total source strength of $\mathbf{F}$ within D, that is,

$$\oiint_{\mathcal{S}} \mathbf{F} \bullet \hat{\mathbf{N}}\, dS = \iiint_D \nabla \bullet \mathbf{F}\, dV.$$

This is the **Divergence Theorem**, which we will prove in Section 16.4.

Example 3 Verify that the vector field $\mathbf{F} = m\mathbf{r}/|\mathbf{r}|^3$, due to a source of strength m at $(0, 0, 0)$, has zero divergence at all points in $\mathbb{R}^3$ except the origin. What would you expect to be the total flux of $\mathbf{F}$ outward across the boundary surface of a domain D if the origin lies outside D? if the origin is inside D?

Solution Since

$$\mathbf{F}(x, y, z) = \frac{m}{r^3}\Big(x\mathbf{i} + y\mathbf{j} + z\mathbf{k}\Big), \quad \text{where} \quad r^2 = x^2 + y^2 + z^2,$$

and since $\partial r/\partial x = x/r$, we have

$$\frac{\partial F_1}{\partial x} = m\frac{\partial}{\partial x}\left(\frac{x}{r^3}\right) = m\frac{r^3 - 3xr^2\left(\frac{x}{r}\right)}{r^6} = m\frac{r^2 - 3x^2}{r^5}.$$

Similarly,

$$\frac{\partial F_2}{\partial y} = m\frac{r^2 - 3y^2}{r^5} \quad \text{and} \quad \frac{\partial F_3}{\partial z} = m\frac{r^2 - 3z^2}{r^5}.$$

Adding these up, we get $\nabla \bullet \mathbf{F}(x, y, z) = 0$ if $r > 0$.

If the origin lies outside the domain D, then the source density of $\mathbf{F}$ in D is zero, so we would expect the total flux of $\mathbf{F}$ out of D to be zero. If the origin lies inside D, then D contains a source of strength m (producing $4\pi m$ cubic units of fluid per unit time), so we would expect the flux out of D to be $4\pi m$. See Example 1 and Exercises 9 and 10 of Section 15.6 for specific examples.

■

Distributions and Delta Functions

If $\ell(x)$ represents the line density (mass per unit length) of mass distributed on the x-axis, then the total mass so distributed is

$$m = \int_{-\infty}^{\infty} \ell(x)\, dx.$$

Now suppose that the only mass on the axis is a "point mass" $m = 1$ located at the origin. Then at all other points $x \neq 0$, the density is $\ell(x) = 0$, but we must still have

$$\int_{-\infty}^{\infty} \ell(x)\, dx = m = 1,$$

so $\ell(0)$ must be infinite. This is an ideal situation, a mathematical model. No real *function* $\ell(x)$ can have such properties; if a function is zero everywhere except at a single point, then any integral of that function will be zero. (Why?) (Also, no real mass can occupy just a single point.) Nevertheless, it is very useful to model real, isolated masses as point masses and to model their densities using **generalized functions** (also called **distributions**).

We can think of the density of a point mass 1 at $x = 0$ as the limit of large densities concentrated on small intervals. For instance, if

$$d_n(x) = \begin{cases} n/2 & \text{if } |x| \le 1/n \\ 0 & \text{if } |x| > 1/n \end{cases}$$

(see Figure 16.1), then for any smooth function $f(x)$ defined on $\mathbb{R}$ we have

$$\int_{-\infty}^{\infty} d_n(x)\, f(x)\, dx = \frac{n}{2} \int_{-1/n}^{1/n} f(x)\, dx.$$

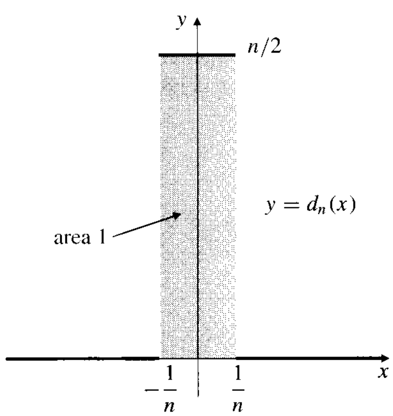

Figure 16.1 The functions $d_n(x)$ converge to $\delta(x)$ as $n \to \infty$

Replace $f(x)$ in the integral on the right with its Maclaurin series:

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \cdots.$$

Since

$$\int_{-1/n}^{1/n} x^k\, dx = \begin{cases} 2/((k+1)n^{k+1}) & \text{if } k \text{ is even} \\ 0 & \text{if } k \text{ is odd,} \end{cases}$$

we can take the limit as $n \to \infty$ and obtain

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} d_n(x)\, f(x)\, dx = f(0).$$

DEFINITION 1

The **Dirac distribution** $\delta(x)$ (also called the **Dirac delta function**, although it is really not a function) is the "limit" of the sequence $d_n(x)$ as $n \to \infty$. It is defined by the requirement that

$$\int_{-\infty}^{\infty} \delta(x) f(x)\, dx = f(0)$$

for every smooth function $f(x)$.

A formal change of variables shows that the delta function also satisfies

$$\int_{-\infty}^{\infty} \delta(x-t) f(t)\, dt = f(x).$$

Example 4 In view of the fact that $\mathbf{F}(\mathbf{r}) = m\mathbf{r}/|\mathbf{r}|^3$ satisfies $\mathbf{div}\,\mathbf{F}(x, y, z) = 0$ for $(x, y, z) \neq (0, 0, 0)$ but produces a flux of $4\pi m$ out of any sphere centred at the origin, we can regard $\mathbf{div}\,\mathbf{F}(x, y, z)$ as a distribution

$$\mathbf{div}\,\mathbf{F}(x, y, z) = 4\pi m \delta(x)\delta(y)\delta(z).$$

In particular, integrating this distribution against $f(x, y, z) = 1$ over $\mathbb{R}^3$, we have

$$\begin{aligned}\iiint_{\mathbb{R}^3} \mathbf{div}\,\mathbf{F}(x, y, z)\, dV &= 4\pi m \int_{-\infty}^{\infty} \delta(x)\, dx \int_{-\infty}^{\infty} \delta(y)\, dy \int_{-\infty}^{\infty} \delta(z)\, dz \\ &= 4\pi m.\end{aligned}$$

The integral can equally well be taken over *any domain* in $\mathbb{R}^3$ that contains the origin in its interior, and the result will be the same. If the origin is outside the domain, the result will be zero. We will reexamine this situation after establishing the Divergence Theorem in Section 16.4. ■

A formal study of distributions is beyond the scope of this book and is usually undertaken in more advanced textbooks on differential equations and engineering mathematics.

Interpretation of the Curl

Roughly speaking, $\mathbf{curl}\,\mathbf{F}(P)$ measures the extent to which the vector field $\mathbf{F}$ "swirls" around P.

Example 5 Consider the velocity field,

$$\mathbf{v} = -\Omega y\mathbf{i} + \Omega x\mathbf{j},$$

of a solid rotating with angular speed Ω about the z-axis, that is, with angular velocity $\boldsymbol{\Omega} = \Omega\mathbf{k}$. (See Figure 15.2 in Section 15.1.) Calculate the circulation of this field around a circle C_ϵ in the xy-plane centred at any point (x_0, y_0), having radius ϵ, and oriented counterclockwise. What is the relationship between this circulation and the curl of $\mathbf{v}$?

Solution The indicated circle has parametrization

$$\mathbf{r} = (x_0 + \epsilon \cos t)\mathbf{i} + (y_0 + \epsilon \sin t)\mathbf{j}, \qquad (0 \le t \le 2\pi),$$

and the circulation of $\mathbf{v}$ around it is given by

$$\oint_{C_\epsilon} \mathbf{v} \bullet d\mathbf{r} = \int_0^{2\pi} \Big(-\Omega(y_0 + \epsilon \sin t)(-\epsilon \sin t) + \Omega(x_0 + \epsilon \cos t)(\epsilon \cos t)\Big)\, dt$$
$$= \int_0^{2\pi} \Big(\Omega\epsilon(y_0 \sin t + x_0 \cos t) + \Omega\epsilon^2\Big)\, dt$$
$$= 2\Omega\pi\epsilon^2.$$

Since

$$\mathbf{curl}\,\mathbf{v} = \nabla\times\mathbf{v} = \Big(\frac{\partial}{\partial x}(\Omega x) - \frac{\partial}{\partial y}(-\Omega y)\Big)\mathbf{k} = 2\Omega\mathbf{k} = 2\boldsymbol{\Omega},$$

the circulation is the product of $(\mathbf{curl}\,\mathbf{v}) \bullet \mathbf{k}$ and the area bounded by $\mathcal{C}_\epsilon$. Note that this circulation is constant for circles of any fixed radius; it does not depend on the position of the centre.

The calculations in the example above suggest that the curl of a vector field is a measure of the *circulation per unit area* in planes normal to the curl. A more precise version of this conjecture is stated in Theorem 2 below. We will not prove this theorem now because a proof at this stage would be quite complicated. (However, see Exercise 14 below for a special case.) A simple proof can be based on Stokes's Theorem. (See Exercise 13 in Section 16.5.)

THEOREM 2

The curl as circulation density

If $\mathbf{F}$ is a smooth vector field and $\mathcal{C}_\epsilon$ is a circle of radius ϵ centred at point P and bounding a disk $\mathcal{S}_\epsilon$ with unit normal $\hat{\mathbf{N}}$ (and orientation inherited from $\mathcal{C}_\epsilon$ — see Figure 16.2), then

$$\lim_{\epsilon\to 0^+} \frac{1}{\pi\epsilon^2} \oint_{\mathcal{C}_\epsilon} \mathbf{F} \bullet d\mathbf{r} = \hat{\mathbf{N}} \bullet \mathbf{curl}\,\mathbf{F}(P).$$

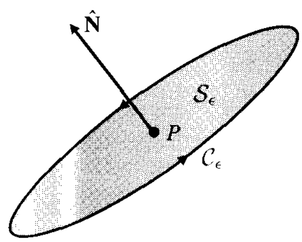

Figure 16.2

Example 5 also suggests the following definition for the *local* angular velocity of a moving fluid:

The **local angular velocity** at point P in a fluid moving with velocity field $\mathbf{v}(P)$ is given by

$$\boldsymbol{\Omega}(P) = \frac{1}{2}\mathbf{curl}\,\mathbf{v}(P).$$

Theorem 2 states that the local angular velocity $\boldsymbol{\Omega}(P)$ is that vector whose component in the direction of any unit vector $\hat{\mathbf{N}}$ is one-half of the limiting circulation per unit area around the (oriented) boundary circles of small disks centred at P and having normal $\hat{\mathbf{N}}$.

Not all vector fields with nonzero curl *appear* to circulate. The velocity field for the rigid body rotation considered in Example 5 appears to circulate around the axis of rotation, but the circulation around a circle in a plane perpendicular to that axis turned out to be independent of the position of the circle; it depended only on its area. The circle need not even surround the axis of rotation. The following example investigates a fluid velocity field whose streamlines are *straight lines* but which still has nonzero, constant curl and, therefore, constant local angular velocity.

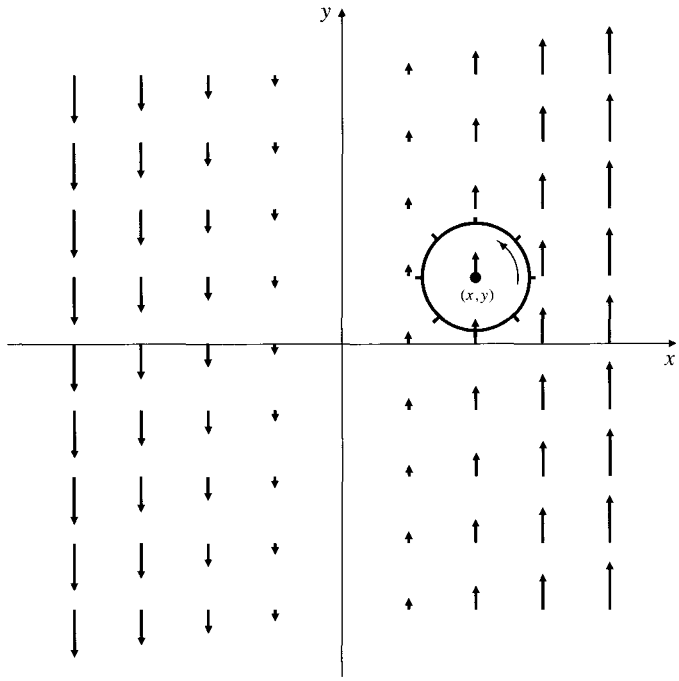

Figure 16.3 The paddle wheel is not only carried along but is set rotating by the flow

Example 6 Consider the velocity field $\mathbf{v} = x\mathbf{j}$ of a fluid moving in the xy-plane. Evidently, particles of fluid are moving along lines parallel to the y-axis. However, $\mathbf{curl}\,\mathbf{v}(x, y) = \mathbf{k}$, and $\boldsymbol{\Omega}(x, y) = \frac{1}{2}\mathbf{k}$. A small "paddle wheel" of radius ϵ placed with its centre at position (x, y) in the fluid (see Figure 16.3) will be carried along with the fluid at velocity $x\mathbf{j}$ but will also be set rotating with angular velocity $\boldsymbol{\Omega}(x, y) = \frac{1}{2}\mathbf{k}$, which is independent of its position. This angular velocity is due to the fact that the velocity of the fluid along the right side of the wheel exceeds that along the left side. ■

Exercises 16.1

In Exercises 1–11, calculate $\mathbf{div}\,\mathbf{F}$ and $\mathbf{curl}\,\mathbf{F}$ for the given vector fields.

1. $\mathbf{F} = x\mathbf{i} + y\mathbf{j}$

2. $\mathbf{F} = y\mathbf{i} + x\mathbf{j}$

3. $\mathbf{F} = y\mathbf{i} + z\mathbf{j} + x\mathbf{k}$

4. $\mathbf{F} = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$

5. $\mathbf{F} = x\mathbf{i} + x\mathbf{k}$

6. $\mathbf{F} = xy^2\mathbf{i} - yz^2\mathbf{j} + zx^2\mathbf{k}$

7. $\mathbf{F} = f(x)\mathbf{i} + g(y)\mathbf{j} + h(z)\mathbf{k}$

8. $\mathbf{F} = f(z)\mathbf{i} - f(z)\mathbf{j}$

9. $\mathbf{F}(r, \theta) = r\mathbf{i} + \sin\theta\mathbf{j}$, where (r, θ) are polar coordinates in the plane

10. $\mathbf{F} = \hat{\mathbf{r}} = \cos\theta\mathbf{i} + \sin\theta\mathbf{j}$

11. $\mathbf{F} = \hat{\boldsymbol{\theta}} = -\sin\theta\mathbf{i} + \cos\theta\mathbf{j}$.

* **12.** Let $\mathbf{F}$ be a smooth, 3-dimensional vector field. If $B_{a,b,c}$ is the surface of the box $-a \le x \le a$, $-b \le y \le b$, $-c \le z \le c$, with outward normal $\hat{\mathbf{N}}$, show that

$$\lim_{a,b,c\to 0^+} \frac{1}{8abc} \oiint_{B_{a,b,c}} \mathbf{F} \bullet \hat{\mathbf{N}}\, dS = \boldsymbol{\nabla} \bullet \mathbf{F}(0, 0, 0).$$

* **13.** Let $\mathbf{F}$ be a smooth 2-dimensional vector field. If $\mathcal{C}_\epsilon$ is the circle of radius ϵ centred at the origin, and $\hat{\mathbf{N}}$ is the unit outward normal to $\mathcal{C}_\epsilon$, show that

$$\lim_{\epsilon\to 0^+} \frac{1}{\pi\epsilon^2} \oint_{\mathcal{C}_\epsilon} \mathbf{F} \bullet \hat{\mathbf{N}}\, ds = \mathbf{div}\,\mathbf{F}(0, 0).$$

* **14.** Prove Theorem 2 in the special case that $\mathcal{C}_\epsilon$ is the circle in the xy-plane with parametrization $x = \epsilon\cos\theta$, $y = \epsilon\sin\theta$, $(0 \le \theta \le 2\pi)$. In this case $\hat{\mathbf{N}} = \mathbf{k}$. *Hint:* expand $\mathbf{F}(x, y, z)$ in a vector Taylor series about the origin as in the proof of Theorem 1, and calculate the circulation of individual terms around $\mathcal{C}_\epsilon$.

16.2 Some Identities Involving Grad, Div, and Curl

There are numerous identities involving the functions

$$\mathbf{grad}\, f(x, y, z) = \nabla f(x, y, z) = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k},$$

$$\mathbf{div}\, \mathbf{F}(x, y, z) = \nabla \bullet \mathbf{F}(x, y, z) = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z},$$

$$\mathbf{curl}\, \mathbf{F}(x, y, z) = \nabla \times \mathbf{F}(x, y, z) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix},$$

and the **Laplacian operator**, $\nabla^2 = \nabla \bullet \nabla$, defined for a scalar field ϕ by

$$\nabla^2 \phi = \nabla \bullet \nabla \phi = \mathbf{div}\,\mathbf{grad}\,\phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2},$$

and for a vector field $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$ by

$$\nabla^2 \mathbf{F} = (\nabla^2 F_1)\mathbf{i} + (\nabla^2 F_2)\mathbf{j} + (\nabla^2 F_3)\mathbf{k}.$$

(The Laplacian operator, $\nabla^2 = (\partial^2/\partial x^2) + (\partial^2/\partial y^2) + (\partial^2/\partial z^2)$, is denoted by Δ in some books.) Recall that a function ϕ is called **harmonic** in a domain D if $\nabla^2 \phi = 0$ throughout D. (See Section 12.4.)

We collect the most important identities together in the following theorem. Most of them are forms of the Product Rule. We will prove a few of the identities to illustrate the techniques involved (mostly brute-force calculation) and leave the rest as exercises. Note that two of the identities involve quantities like $(\mathbf{G} \bullet \nabla)\mathbf{F}$; this represents the vector obtained by applying the scalar differential operator $\mathbf{G} \bullet \nabla$ to the vector field $\mathbf{F}$:

$$(\mathbf{G} \bullet \nabla)\mathbf{F} = G_1 \frac{\partial \mathbf{F}}{\partial x} + G_2 \frac{\partial \mathbf{F}}{\partial y} + G_3 \frac{\partial \mathbf{F}}{\partial z}.$$

THEOREM 3 **Vector differential identities**

Let ϕ and ψ be scalar fields and $\mathbf{F}$ and $\mathbf{G}$ be vector fields, all assumed to be sufficiently smooth that all the partial derivatives in the identities are continuous. Then the following identities hold:

(a) $\nabla(\phi\psi) = \phi\nabla\psi + \psi\nabla\phi$

(b) $\nabla \bullet (\phi\mathbf{F}) = (\nabla\phi) \bullet \mathbf{F} + \phi(\nabla \bullet \mathbf{F})$

(c) $\nabla \times (\phi\mathbf{F}) = (\nabla\phi) \times \mathbf{F} + \phi(\nabla \times \mathbf{F})$

(d) $\nabla \bullet (\mathbf{F} \times \mathbf{G}) = (\nabla \times \mathbf{F}) \bullet \mathbf{G} - \mathbf{F} \bullet (\nabla \times \mathbf{G})$

(e) $\nabla \times (\mathbf{F} \times \mathbf{G}) = (\nabla \bullet \mathbf{G})\mathbf{F} + (\mathbf{G} \bullet \nabla)\mathbf{F} - (\nabla \bullet \mathbf{F})\mathbf{G} - (\mathbf{F} \bullet \nabla)\mathbf{G}$

(f) $\nabla(\mathbf{F} \bullet \mathbf{G}) = \mathbf{F}\times(\nabla\times\mathbf{G}) + \mathbf{G}\times(\nabla\times\mathbf{F}) + (\mathbf{F} \bullet \nabla)\mathbf{G} + (\mathbf{G} \bullet \nabla)\mathbf{F}$

(g) $\nabla \bullet (\nabla\times\mathbf{F}) = 0$ (**div curl** $= 0$)

(h) $\nabla\times(\nabla\phi) = \mathbf{0}$ (**curl grad** $= \mathbf{0}$)

(i) $\nabla\times(\nabla\times\mathbf{F}) = \nabla(\nabla \bullet \mathbf{F}) - \nabla^2\mathbf{F}$

(**curl curl** $=$ **grad div** $-$ Laplacian)

Identities (a)–(f) are versions of the Product Rule and are first-order identities involving only one application of ∇. Identities (g)–(i) are second-order identities. Identities (g) and (h) are equivalent to the equality of mixed partial derivatives and are especially important for the understanding of **div** and **curl**.

PROOF We will prove only identities (c), (e), and (g). The remaining proofs are similar to these.

(c) The first component (**i** component) of $\nabla\times(\phi\mathbf{F})$ is

$$\frac{\partial}{\partial y}(\phi F_3) - \frac{\partial}{\partial z}(\phi F_2) = \frac{\partial\phi}{\partial y}F_3 - \frac{\partial\phi}{\partial z}F_2 + \phi\frac{\partial F_3}{\partial y} - \phi\frac{\partial F_2}{\partial z}.$$

The first two terms on the right constitute the first component of $(\nabla\phi)\times\mathbf{F}$, and the last two terms constitute the first component of $\phi(\nabla\times\mathbf{F})$. Therefore, the first components of both sides of identity (c) are equal. The equality of the other components follows similarly.

(e) Again, it is sufficient to show that the first components of the vectors on both sides of the identity are equal. To calculate the first component of $\nabla\times(\mathbf{F}\times\mathbf{G})$ we need the second and third components of $\mathbf{F}\times\mathbf{G}$, which are

$$(\mathbf{F}\times\mathbf{G})_2 = F_3G_1 - F_1G_3 \quad \text{and} \quad (\mathbf{F}\times\mathbf{G})_3 = F_1G_2 - F_2G_1.$$

The first component of $\nabla\times(\mathbf{F}\times\mathbf{G})$ is therefore

$$\begin{aligned}
&\frac{\partial}{\partial y}(F_1G_2 - F_2G_1) - \frac{\partial}{\partial z}(F_3G_1 - F_1G_3)\\
&\qquad = \frac{\partial F_1}{\partial y}G_2 + F_1\frac{\partial G_2}{\partial y} - \frac{\partial F_2}{\partial y}G_1 - F_2\frac{\partial G_1}{\partial y} - \frac{\partial F_3}{\partial z}G_1\\
&\qquad\quad - F_3\frac{\partial G_1}{\partial z} + \frac{\partial F_1}{\partial z}G_3 + F_1\frac{\partial G_3}{\partial z}.
\end{aligned}$$

The first components of the four terms on the right side of identity (e) are

$$\begin{aligned}
((\nabla \bullet \mathbf{G})\mathbf{F})_1 &= F_1\frac{\partial G_1}{\partial x} + F_1\frac{\partial G_2}{\partial y} + F_1\frac{\partial G_3}{\partial z}\\
((\mathbf{G} \bullet \nabla)\mathbf{F})_1 &= \frac{\partial F_1}{\partial x}G_1 + \frac{\partial F_1}{\partial y}G_2 + \frac{\partial F_1}{\partial z}G_3\\
-((\nabla \bullet \mathbf{F})\mathbf{G})_1 &= -\frac{\partial F_1}{\partial x}G_1 - \frac{\partial F_2}{\partial y}G_1 - \frac{\partial F_3}{\partial z}G_1\\
-((\mathbf{F} \bullet \nabla)\mathbf{G})_1 &= -F_1\frac{\partial G_1}{\partial x} - F_2\frac{\partial G_1}{\partial y} - F_3\frac{\partial G_1}{\partial z}.
\end{aligned}$$

(g) This is a straightforward calculation involving the equality of mixed partial derivatives:

$$\begin{aligned}\nabla \bullet (\nabla \times \mathbf{F}) &= \frac{\partial}{\partial x}\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right) \\ &\quad + \frac{\partial}{\partial z}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) \\ &= \frac{\partial^2 F_3}{\partial x \partial y} - \frac{\partial^2 F_2}{\partial x \partial z} + \frac{\partial^2 F_1}{\partial y \partial z} - \frac{\partial^2 F_3}{\partial y \partial x} + \frac{\partial^2 F_2}{\partial z \partial x} - \frac{\partial^2 F_1}{\partial z \partial y} \\ &= 0.\end{aligned}$$

Remark Two *triple product* identities for vectors were previously presented in Exercises 18 and 23 of Section 10.3:

$$\mathbf{a} \bullet (\mathbf{b} \times \mathbf{c}) = \mathbf{b} \bullet (\mathbf{c} \times \mathbf{a}) = \mathbf{c} \bullet (\mathbf{a} \times \mathbf{b}),$$

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \bullet \mathbf{c})\mathbf{b} - (\mathbf{a} \bullet \mathbf{b})\mathbf{c}.$$

While these are useful identities, they *cannot* be used to give simpler proofs of the identities in Theorem 3 by replacing one or other of the vectors with ∇. (Why?)

Scalar and Vector Potentials

Two special terms are used to describe vector fields for which either the divergence or the curl is identically zero.

DEFINITION 2

Solenoidal and irrotational vector fields

A vector field $\mathbf{F}$ is called **solenoidal** in a domain D if $\mathbf{div}\,\mathbf{F} = 0$ in D.
A vector field $\mathbf{F}$ is called **irrotational** in a domain D if $\mathbf{curl}\,\mathbf{F} = \mathbf{0}$ in D.

Part (h) of Theorem 3 says that $\mathbf{F} = \mathbf{grad}\,\phi \Longrightarrow \mathbf{curl}\,\mathbf{F} = \mathbf{0}$. Thus,

Every conservative vector field is irrotational.

Part (g) of Theorem 3 says that $\mathbf{F} = \mathbf{curl}\,\mathbf{G} \Longrightarrow \mathbf{div}\,\mathbf{F} = 0$. Thus,

The curl of any vector field is solenoidal.

The *converses* of these assertions hold if the domain of $\mathbf{F}$ satisfies certain conditions.

THEOREM 4

If $\mathbf{F}$ is a smooth, irrotational vector field on a simply connected domain D, then $\mathbf{F} = \nabla\phi$ for some scalar potential function defined on D, so $\mathbf{F}$ is conservative.

THEOREM 5

If $\mathbf{F}$ is a smooth, solenoidal vector field on a domain D with the property that every closed surface in D bounds a domain contained in D, then $\mathbf{F} = \mathbf{curl}\,\mathbf{G}$ for some vector field $\mathbf{G}$ defined on D. Such a vector field $\mathbf{G}$ is called a **vector potential** of the vector field $\mathbf{F}$.

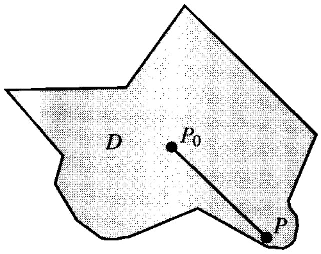

Figure 16.4 The line segment from P_0 to any point in D lies in D

We cannot prove these results in their full generality at this point. However, both theorems have simple proofs in the special case where the domain D is **star-like**. A star-like domain is one for which there exists a point P_0 such that the line segment from P_0 to any point P in D lies wholly in D. (See Figure 16.4.) Both proofs are *constructive* in that they tell you how to find a potential.

Proof of Theorem 4 for star-like domains. Without loss of generality, we can assume that P_0 is the origin. If $P = (x, y, z)$ is any point in D, then the straight line segment

$$\mathbf{r}(t) = tx\mathbf{i} + ty\mathbf{j} + tz\mathbf{k}, \qquad (0 \le t \le 1),$$

from P_0 to P lies in D. Define the function ϕ on D by

$$\begin{aligned}\phi(x, y, z) &= \int_0^1 \mathbf{F}\big(\mathbf{r}(t)\big) \bullet \frac{d\mathbf{r}}{dt}\, dt \\ &= \int_0^1 \Big(xF_1(\xi, \eta, \zeta) + yF_2(\xi, \eta, \zeta) + zF_3(\xi, \eta, \zeta)\Big)\, dt,\end{aligned}$$

where $\xi = tx$, $\eta = ty$, and $\zeta = tz$. We calculate $\partial\phi/\partial x$, making use of the fact that $\mathbf{curl}\,\mathbf{F} = \mathbf{0}$ to replace $(\partial/\partial\xi)F_2(\xi, \eta, \zeta)$ with $(\partial/\partial\eta)F_1(\xi, \eta, \zeta)$ and $(\partial/\partial\xi)F_3(\xi, \eta, \zeta)$ with $(\partial/\partial\zeta)F_1(\xi, \eta, \zeta)$:

$$\begin{aligned}\frac{\partial\phi}{\partial x} &= \int_0^1 \left(F_1(\xi, \eta, \zeta) + tx\frac{\partial F_1}{\partial\xi} + ty\frac{\partial F_2}{\partial\xi} + tz\frac{\partial F_3}{\partial\xi}\right) dt \\ &= \int_0^1 \left(F_1(\xi, \eta, \zeta) + tx\frac{\partial F_1}{\partial\xi} + ty\frac{\partial F_1}{\partial\eta} + tz\frac{\partial F_1}{\partial\zeta}\right) dt \\ &= \int_0^1 \frac{d}{dt}\Big(t\,F_1(\xi, \eta, \zeta)\Big)\, dt \\ &= \Big(t\,F_1(tx, ty, tz)\Big)\Big|_0^1 = F_1(x, y, z).\end{aligned}$$

Similarly, $\partial\phi/\partial y = F_2$ and $\partial\phi/\partial z = F_3$. Thus $\nabla\phi = \mathbf{F}$.

The details of the proof of Theorem 5 for star-like domains are similar to those of Theorem 4, and we relegate the proof to Exercise 18 below.

Note that vector potentials, when they exist, are *very* nonunique. Since **curl grad** ϕ is identically zero (Theorem 3(h)), an arbitrary conservative field can be added to **G** without changing the value of **curl G**. The following example illustrates just how much freedom you have in making simplifying assumptions when trying to find a vector potential.

Example 1 Show that the vector field $\mathbf{F} = (x^2+yz)\mathbf{i} - 2y(x+z)\mathbf{j} + (xy+z^2)\mathbf{k}$ is solenoidal in $\mathbb{R}^3$ and find a vector potential for it.

Solution Since $\mathbf{div}\,\mathbf{F} = 2x - 2(x+z) + 2z = 0$ in $\mathbb{R}^3$, $\mathbf{F}$ is solenoidal. A vector potential $\mathbf{G}$ for $\mathbf{F}$ must satisfy $\mathbf{curl}\,\mathbf{G} = \mathbf{F}$, that is,

$$\frac{\partial G_3}{\partial y} - \frac{\partial G_2}{\partial z} = x^2 + yz,$$
$$\frac{\partial G_1}{\partial z} - \frac{\partial G_3}{\partial x} = -2xy - 2yz,$$
$$\frac{\partial G_2}{\partial x} - \frac{\partial G_1}{\partial y} = xy + z^2.$$

The three components of $\mathbf{G}$ have nine independent first partial derivatives, so there are nine "degrees of freedom" involved in their determination. The three equations above use up three of these nine degrees of freedom. That leaves six. Let us try to find a solution $\mathbf{G}$ with $G_2 = 0$ identically. This means that all three first partials of G_2 are zero, so we have used up three degrees of freedom in making this assumption. We have three left. The first equation now implies that

$$G_3 = \int (x^2 + yz)\,dy = x^2y + \frac{1}{2}y^2z + M(x, z).$$

(Since we were integrating with respect to y, the constant of integration can still depend on x and z.) We make a second simplifying assumption, that $M(x, z) = 0$. This uses up two more degrees of freedom, leaving one. From the second equation we have

$$\frac{\partial G_1}{\partial z} = \frac{\partial G_3}{\partial x} - 2xy - 2yz = 2xy - 2xy - 2yz = -2yz,$$

so

$$G_1 = -2\int yz\,dz = -yz^2 + N(x, y).$$

We cannot assume that $N(x, y) = 0$ identically because that would require two degrees of freedom and we have only one. However, the third equation implies

$$xy + z^2 = -\frac{\partial G_1}{\partial y} = z^2 - \frac{\partial N}{\partial y}.$$

Thus, $(\partial/\partial y)N(x, y) = -xy$; observe that the terms involving z have cancelled out. This happened because $\mathbf{div}\,\mathbf{F} = 0$. Had $\mathbf{F}$ not been solenoidal, we could not have determined N as a function of x and y only from the above equation. As it is, however, we have

$$N(x, y) = -\int xy\,dy = -\frac{1}{2}xy^2 + P(x).$$

We can use our last degree of freedom to choose $P(x)$ to be identically zero and hence obtain

$$\mathbf{G} = -\left(yz^2 + \frac{xy^2}{2}\right)\mathbf{i} + \left(x^2y + \frac{y^2z}{2}\right)\mathbf{k}$$

as the required vector potential for $\mathbf{F}$. You can check that $\mathbf{curl}\,\mathbf{G} = \mathbf{F}$. Of course, other choices of simplifying assumptions would have led to very different functions $\mathbf{G}$, which would have been equally correct. ■

Maple Calculations

The Maple **linalg** package defines routines **grad**, **laplacian**, **diverge**, and **curl** that calculate the gradient and laplacian of scalar expressions and the divergence and curl of vector expressions with respect to a given vector of variables.

```
> with(linalg):  xyz := [x,y,z];
         grad(exp(x)*sin(y)+3*z,xyz);
```

$$xyz := [x, y, z]$$

$$[e^x \sin(y), e^x \cos(y), 3]$$

```
> v := [x*y, y*z, z*x]; diverge(v,xyz);
```

$$v := [xy, yz, zx]$$

$$y + z + x$$

```
> curlv := curl(v,xyz);
```

$$curlv := [-y, -z, -x]$$

Because grad, diverge, and curl just convert expressions to expressions, it is awkward to evaluate the output at a specific point. For instance, we can't obtain the value of **curl v**$(1, 2, 3)$ by entering `curlv(1,2,3)`:

```
> curlv(1,2,3);
```

$$[-y(1, 2, 3), -z(1, 2, 3), -x(1, 2, 3)]$$

Instead, we have to evaluate `curlv` as a vector with `evalm` and then use `subs` to substitute the values for x, y, and z into the result.

```
> subs(x=1,y=2,z=3,evalm(curlv));
```

$$[-2, -3, -1]$$

In order to simplify the use of the vector differential operators, let us define new routines, which we will call `Grad`, `Div`, and `Curl`, so that each is an *operator* converting functions to functions, rather than expressions to expressions. In addition, we will define vector and scalar versions of the Laplace operator, `lapl`, and `Lapl`, an operator `&dotDel` for calculating $(\mathbf{F} \bullet \nabla)\mathbf{G}$, and a procedure `MakeVecFcn` to facilitate the definition of vector-valued functions. These definitions are collected in the file **vecdiff.def**, which we list here. It can be found on the author's website (`www.pearsoned.ca/text/adams_calc`).

```
MakeVecFcn := proc(u,v,w)
   [unapply(u,x,y,z),unapply(v,x,y,z),unapply(w,x,y,z)]
   end;
Grad := F -> [D[1](F),D[2](F),D[3](F)];
Div := F -> D[1](F[1])+D[2](F[2])+D[3](F[3]);
Curl := F -> [D[2](F[3])-D[3](F[2]),
   D[3](F[1])-D[1](F[3]),D[1](F[2])-D[2](F[1])];
lapl := F -> D[1,1](F)+D[2,2](F)+D[3,3](F);
Lapl := F -> [lapl(F[1]),lapl(F[2]),lapl(F[3])];
'&dotDel' := proc(U,V)
   options operator, arrow; local i, j;
   [seq(sum(U[i]*D[i](V[j]),i=1..3),j=1..3)]
   end;
```

These definitions can be read in to a Maple session with the command

```
> read "vecdiff.def";
```

In addition, we will want to read in the definitions in the file **vecops.def** discussed in Section 10.7. (This file in turn loads the **linalg** package.)

```
> read "vecops.def";
```

Now let us illustrate the usage of the procedures defined in `vecdiff.def`. It should be noted that all vectors used with these procedures must be 3-vectors, and functions should depend on (at most) three variables. `Grad` takes a scalar function to a vector function. In Maple, an undefined name can represent an (arbitrary) scalar function:

```
> Grad(g);
```

$$[D_1(g), D_2(g), D_3(g)]$$

A vector-valued general function must, however, be declared to be a vector:

```
> U := vector(3); Curl(U);
```

$$U := array(1..3, [])$$

$$[D_2(U_3) - D_3(U_2), D_3(U_1) - D_1(U_3), D_1(U_2) - D_2(U_1)]$$

Explicitly defined scalar-valued functions can be defined using the usual "->" notation:

```
> f := (x,y,z) -> x^2*y + y^2*z^3;
```

$$f := (x, y, z) \to x^2y + y^2z^3$$

```
> Grad(f); Grad(f)(1,2,3);
```

$$[(x, y, z) \to 2xy, (x, y, z) \to x^2 + 2yz^3, (x, y, z) \to 3y^2z^2]$$

$$[4, 109, 108]$$

However, we cannot use this technique to define a vector-valued function. If we try to use, say,

```
> F := (x,y,z) -> [x^2*y, y^3*z, z^4*x];
```

we will get a function whose values are the desired vectors, but Maple will not regard the function F itself as being a vector with component scalar functions `F[1]`, `F[2]`, and `F[3]`. To get such a "vector function" we can feed the desired components to the procedure `MakeVecFcn` defined in `vecdiff.def`.

```
> F := MakeVecFcn(x^2*y,y^3*z,z^4*x);
```

$$F := [(x, y, z) \to x^2y, (x, y, z) \to y^3z, (x, y, z) \to z^4x]$$

NOTE: `MakeVecFcn` only defines three-dimensional vector functions, and the three component expressions u, v, and w will be considered to be expressions in the variables x, y, and z. These variables must not have been assigned any values or the procedure won't work.

```
> Div(F)(3,-2,1); Curl(F)(a,b,c);
```

$$12$$

$$[-b^3, -c^4, -a^2]$$

```
> Div(Curl(F))(1,2,3); Curl(Grad(f))(a,b,c);
```

$$0$$

$$[0, 0, 0]$$

Of course, `Div(Curl(U))` should be 0 for any three-dimensional field **U**, and `Curl(Grad(g)` should be [0, 0, 0] for any scalar field g. We can verify these for our general (unspecified) fields `U` and `g` just as easily.

```
> Div(Curl(U)); Curl(Grad(g));
```

$$0$$

$$[0, 0, 0]$$

Here are some calculations involving the scalar and vector Laplacian operators:

```
> lapl(f)(a,b,c); Lapl(F)(a,b,c); lapl(g)(a,b,c);
```

$$2b + 2c^3 + 6b^2c$$

$$[2b, 6bc, 12c^2a]$$

$$D_{1,1}(g)(a,b,c) + D_{2,2}(g)(a,b,c) + D_{3,3}(g)(a,b,c)$$

We can verify the identity **curl curl U** = **grad div U** − Laplacian(**U**) by subtracting the right side from the left:

```
> Curl(Curl(U))-Grad(Div(U))+Lapl(U);
```

$$[0, 0, 0]$$

Other identities can be verified similarly. Sometimes some simplification of the result is necessary. You can use `simplify(%)` to simplify an immediately previous scalar result, or `evl(%)` (defined in **vecops.def**) to simplify the immediately previous vector result.

As a final example let us verify the identity

$$\nabla(\mathbf{U} \bullet \mathbf{V}) = \mathbf{U}\times(\nabla\times\mathbf{V}) + \mathbf{V}\times(\nabla\times\mathbf{U}) + (\mathbf{U} \bullet \nabla)\mathbf{V} + (\mathbf{V} \bullet \nabla)\mathbf{U}.$$

```
> V := vector(3);
     Grad(U &.  V) - U &x Curl(V) - V &x Curl(U)
     - U &dotDel V - V &dotDel U;
```

This input produces several lines of output: an unsimplified vector involving various combinations of the components of **U** and **V** and their first partial derivatives. To simplify it, we use the `evl` routine from `vecops.def`.

```
> evl(%);
```

$$[0, 0, 0]$$

which confirms the identity.

Exercises 16.2

1. Prove part (a) of Theorem 3.
2. Prove part (b) of Theorem 3.
3. Prove part (d) of Theorem 3.
4. Prove part (f) of Theorem 3.
5. Prove part (h) of Theorem 3.
6. Prove part (i) of Theorem 3.
7. * Given that the field lines of the vector field $\mathbf{F}(x, y, z)$ are parallel straight lines, can you conclude anything about **div F**? about **curl F**?
8. Let $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and let **c** be a constant vector. Show that $\nabla \bullet (\mathbf{c}\times\mathbf{r}) = 0$, $\nabla\times(\mathbf{c}\times\mathbf{r}) = 2\mathbf{c}$, and $\nabla(\mathbf{c} \bullet \mathbf{r}) = \mathbf{c}$.
9. Let $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and let $r = |\mathbf{r}|$. If f is a differentiable function of one variable, show that
$$\nabla \bullet (f(r)\mathbf{r}) = rf'(r) + 3f(r).$$
Find $f(r)$ if $f(r)\mathbf{r}$ is solenoidal for $r \neq 0$.
10. If the smooth vector field **F** is both irrotational and solenoidal on $\mathbb{R}^3$, show that the three components of **F** and the scalar potential for **F** are all harmonic functions in $\mathbb{R}^3$.

11. If $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and $\mathbf{F}$ is smooth, show that

$$\nabla\times(\mathbf{F}\times\mathbf{r}) = \mathbf{F} - (\nabla\bullet\mathbf{F})\mathbf{r} + \nabla(\mathbf{F}\bullet\mathbf{r}) - \mathbf{r}\times(\nabla\times\mathbf{F}).$$

In particular, if $\nabla\bullet\mathbf{F} = 0$ and $\nabla\times\mathbf{F} = \mathbf{0}$, then

$$\nabla\times(\mathbf{F}\times\mathbf{r}) = \mathbf{F} + \nabla(\mathbf{F}\bullet\mathbf{r}).$$

12. If ϕ and ψ are harmonic functions, show that $\phi\nabla\psi - \psi\nabla\phi$ is solenoidal.

13. If ϕ and ψ are smooth scalar fields, show that

$$\nabla\times(\phi\nabla\psi) = -\nabla\times(\psi\nabla\phi) = \nabla\phi\times\nabla\psi.$$

14. Verify the identity

$$\nabla\bullet\Big(f(\nabla g\times\nabla h)\Big) = \nabla f\bullet(\nabla g\times\nabla h)$$

for smooth scalar fields f, g, and h.

15. If the vector fields $\mathbf{F}$ and $\mathbf{G}$ are smooth and conservative, show that $\mathbf{F}\times\mathbf{G}$ is solenoidal. Find a vector potential for $\mathbf{F}\times\mathbf{G}$.

16. Find a vector potential for $\mathbf{F} = -y\mathbf{i} + x\mathbf{j}$.

17. Show that $\mathbf{F} = xe^{2z}\mathbf{i} + ye^{2z}\mathbf{j} - e^{2z}\mathbf{k}$ is a solenoidal vector field, and find a vector potential for it.

* **18.** Suppose $\mathbf{div}\,\mathbf{F} = 0$ in a domain D any point P of which can by joined to the origin by a straight line segment in D. Let $\mathbf{r} = tx\mathbf{i} + ty\mathbf{j} + tz\mathbf{k}$, $(0 \le t \le 1)$, be a parametrization of the line segment from the origin to (x, y, z) in D. If

$$\mathbf{G}(x, y, z) = \int_0^1 t\mathbf{F}(\mathbf{r}(t))\times\frac{d\mathbf{r}}{dt}\,dt,$$

show that $\mathbf{curl}\,\mathbf{G} = \mathbf{F}$ throughout D. *Hint:* it is enough to check the first components of $\mathbf{curl}\,\mathbf{G}$ and $\mathbf{F}$. Proceed in a manner similar to the proof of Theorem 4.

19. Use Maple and the definitions in the files **vecops.def** and **vecdiff.def** to verify the identities (a)–(e) of Theorem 3.

16.3 Green's Theorem in the Plane

The Fundamental Theorem of Calculus,

$$\int_a^b \frac{d}{dx} f(x)\,dx = f(b) - f(a),$$

expresses the integral, taken over the interval $[a, b]$, of the derivative of a single-variable function, f, as a *sum* of values of that function at the *oriented boundary* of the interval $[a, b]$, that is, at the two endpoints a and b, the former providing a *negative* contribution and the latter a *positive* one. The line integral of a conservative vector field over a curve $\mathcal{C}$ from A to B,

$$\int_{\mathcal{C}} \nabla\phi\bullet d\mathbf{r} = \phi(B) - \phi(A),$$

has a similar interpretation; $\nabla\phi$ is a derivative, and the curve $\mathcal{C}$, although lying in a two- or three-dimensional space, is intrinsically a one-dimensional object, and the points A and B constitute its boundary.

Green's Theorem is a two-dimensional version of the Fundamental Theorem of Calculus that expresses the *double integral* of a certain kind of derivative of a two-dimensional vector field $\mathbf{F}(x, y)$, namely the $\mathbf{k}$-component of $\mathbf{curl}\,\mathbf{F}$, over a region R in the xy-plane as a line integral (i.e., a "sum") of the tangential component of $\mathbf{F}$ around the curve $\mathcal{C}$ which is the oriented boundary of R:

$$\iint_R \mathbf{curl}\,\mathbf{F}\bullet\mathbf{k}\,dA = \oint_{\mathcal{C}} \mathbf{F}\bullet d\mathbf{r},$$

or, more explicitly,

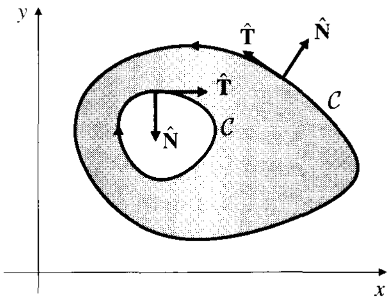

Figure 16.5 A plane domain with positively oriented boundary

$$\iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx\,dy = \oint_C F_1(x, y)\,dx + F_2(x, y)\,dy.$$

For this formula to hold, C must be the oriented boundary of R considered as a surface with orientation provided by $\hat{\mathbf{N}} = \mathbf{k}$. Thus, C is oriented with R on the left as we move around C in the direction of its orientation. We will call such a curve positively oriented with respect to R. In particular, if C is a simple closed curve bounding R, then C is oriented counterclockwise. Of course, R may have holes, and the boundaries of the holes will be oriented clockwise. In any case, the unit tangent $\hat{\mathbf{T}}$ and unit exterior (pointing out of R) normal $\hat{\mathbf{N}}$ on C satisfy $\hat{\mathbf{N}} = \hat{\mathbf{T}} \times \mathbf{k}$. See Figure 16.5.

THEOREM 6

Green's Theorem

Let R be a regular, closed region in the xy-plane whose boundary, C, consists of one or more piecewise smooth, non-self-intersecting, closed curves that are positively oriented with respect to R. If $\mathbf{F} = F_1(x, y)\mathbf{i} + F_2(x, y)\mathbf{j}$ is a smooth vector field on R, then

$$\oint_C F_1(x, y)\,dx + F_2(x, y)\,dy = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dA.$$

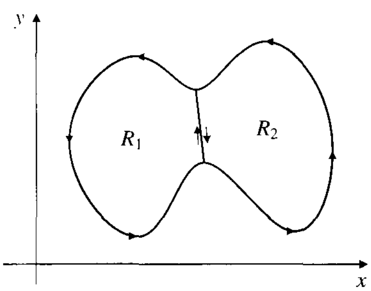

Figure 16.6 Green's Theorem holds for the union of R_1 and R_2 if it holds for each of those regions

PROOF Recall that a regular region can be divided into nonoverlapping subregions that are both x-simple and y-simple. (See Section 14.2.) When two such regions share a common boundary curve, they induce opposite orientations on that curve, so the sum of the line integrals over the boundaries of the subregions is just the line integral over the boundary of the whole region. (See Figure 16.6.) The double integrals over the subregions also add to give the double integral over the whole region. It therefore suffices to show that the formula holds for a region R that is both x-simple and y-simple.

Since R is y-simple, it is specified by inequalities of the form $a \le x \le b$, $f(x) \le y \le g(x)$, with the bottom boundary $y = f(x)$ oriented left to right and the upper boundary $y = g(x)$ oriented right to left. (See Figure 16.7.) Thus,

$$\begin{aligned} -\iint_R \frac{\partial F_1}{\partial y}\,dx\,dy &= -\int_a^b dx \int_{f(x)}^{g(x)} \frac{\partial F_1}{\partial y}\,dy \\ &= \int_a^b \Big(-F_1\big(x, g(x)\big) + F_1\big(x, f(x)\big)\Big)\,dx. \end{aligned}$$

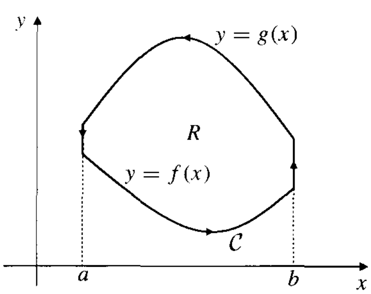

Figure 16.7
$\oint_C F_1\,dx = -\iint_R \frac{\partial F_1}{\partial y}\,dA$ for this y-simple region R

On the other hand, since $dx = 0$ on the vertical sides of R, and the top boundary is traversed from b to a, we have

$$\oint_C F_1(x, y)\,dx = \int_a^b \Big(F_1\big(x, f(x)\big) - F_1\big(x, g(x)\big)\Big)\,dx = \iint_R -\frac{\partial F_1}{\partial y}\,dx\,dy.$$

Similarly, since R is x-simple, $\oint_C F_2\,dy = \iint_R \frac{\partial F_2}{\partial x}\,dx\,dy$, so

$$\oint_C F_1(x, y)\,dx + F_2(x, y)\,dy = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dA.$$

Example 1 **(Area bounded by a simple closed curve)** For any of the three vector fields

$$\mathbf{F} = x\mathbf{j}, \qquad \mathbf{F} = -y\mathbf{i}, \qquad \text{and} \qquad \mathbf{F} = \frac{1}{2}(-y\mathbf{i} + x\mathbf{j}),$$

we have $(\partial F_2/\partial x) - (\partial F_1/\partial y) = 1$. If C is a positively oriented, piecewise smooth, simple closed curve bounding a region R in the plane, then, by Green's Theorem,

$$\oint_C x\,dy = -\oint_C y\,dx = \frac{1}{2}\oint_C x\,dy - y\,dx = \iint_R 1\,dA = \text{area of } R.$$

Example 2 Evaluate $I = \oint_C (x-y^3)\,dx + (y^3+x^3)\,dy$, where C is the positively oriented boundary of the quarter-disk Q: $0 \le x^2 + y^2 \le a^2$, $x \ge 0$, $y \ge 0$.

Solution We use Green's Theorem to calculate I:

$$\begin{aligned} I &= \iint_Q \left(\frac{\partial}{\partial x}(y^3 + x^3) - \frac{\partial}{\partial y}(x - y^3)\right) dA \\ &= 3\iint_Q (x^2 + y^2)\,dA = 3\int_0^{\pi/2} d\theta \int_0^a r^3\,dr = \frac{3}{8}\pi a^4. \end{aligned}$$

Example 3 Let C be a positively oriented, simple closed curve in the xy-plane, bounding a region R and not passing through the origin. Show that

$$\oint_C \frac{-y\,dx + x\,dy}{x^2 + y^2} = \begin{cases} 0 & \text{if the origin is outside } R \\ 2\pi & \text{if the origin is inside } R. \end{cases}$$

Solution First, if $(x, y) \neq (0, 0)$, then, by direct calculation,

$$\frac{\partial}{\partial x}\left(\frac{x}{x^2 + y^2}\right) - \frac{\partial}{\partial y}\left(\frac{-y}{x^2 + y^2}\right) = 0.$$

If the origin is not in R, then Green's Theorem implies that

$$\oint_C \frac{-y\,dx + x\,dy}{x^2 + y^2} = \iint_R \left[\frac{\partial}{\partial x}\left(\frac{x}{x^2 + y^2}\right) - \frac{\partial}{\partial y}\left(\frac{-y}{x^2 + y^2}\right)\right] dx\,dy = 0.$$

Now suppose the origin is in R. Since it is assumed that the origin is not on C, it must be an interior point of R. The interior of R is open, so there exists $\epsilon > 0$ such that the circle C_ϵ of radius ϵ centred at the origin is in the interior of R. Let C_ϵ be oriented negatively (clockwise). By direct calculation (see Exercise 22(a) of Section 15.4) it is easily shown that

$$\oint_{C_\epsilon} \frac{-y\,dx + x\,dy}{x^2 + y^2} = -2\pi.$$

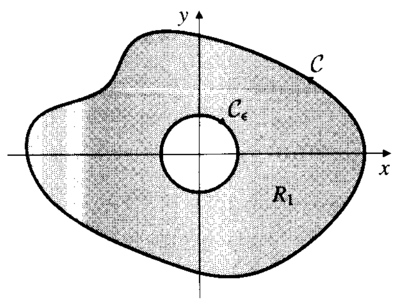

Figure 16.8

Together C and C_ϵ form the positively oriented boundary of a region R_1 that excludes the origin. (See Figure 16.8.) So, by Green's Theorem,

$$\oint_C \frac{-y\,dx + x\,dy}{x^2+y^2} + \oint_{C_\epsilon} \frac{-y\,dx + x\,dy}{x^2+y^2} = 0.$$

The desired result now follows:

$$\oint_C \frac{-y\,dx + x\,dy}{x^2+y^2} = -\oint_{C_\epsilon} \frac{-y\,dx + x\,dy}{x^2+y^2} = -(-2\pi) = 2\pi.$$

The Two-Dimensional Divergence Theorem

The following theorem is an alternative formulation of the two-dimensional Fundamental Theorem of Calculus. In this case we express the double integral of **div F** (a derivative of **F**) over R as a single integral of the outward normal component of **F** on the boundary C of R.

THEOREM 7

The Divergence Theorem in the Plane

Let R be a regular, closed region in the xy-plane whose boundary, C, consists of one or more piecewise smooth, non-self-intersecting, closed curves. Let $\hat{\mathbf{N}}$ denote the unit outward (from R) normal field on C. If $\mathbf{F} = F_1(x,y)\mathbf{i} + F_2(x,y)\mathbf{j}$ is a smooth vector field on R, then

$$\iint_R \operatorname{div} \mathbf{F}\, dA = \oint_C \mathbf{F} \bullet \hat{\mathbf{N}}\, ds.$$

PROOF As observed in the second paragraph of this section, $\hat{\mathbf{N}} = \hat{\mathbf{T}} \times \mathbf{k}$, where $\hat{\mathbf{T}}$ is the unit tangent field in the positive direction on C. If $\hat{\mathbf{T}} = T_1\mathbf{i} + T_2\mathbf{j}$, then $\hat{\mathbf{N}} = T_2\mathbf{i} - T_1\mathbf{j}$. (See Figure 16.9.) Now let **G** be the vector field with components $G_1 = -F_2$ and $G_2 = F_1$. Then $\mathbf{G} \bullet \hat{\mathbf{T}} = \mathbf{F} \bullet \hat{\mathbf{N}}$ and, by Green's Theorem,

$$\begin{aligned}\iint_R \operatorname{div} \mathbf{F}\, dA &= \iint_R \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y}\right) dA \\ &= \iint_R \left(\frac{\partial G_2}{\partial x} - \frac{\partial G_1}{\partial y}\right) dA \\ &= \oint_C \mathbf{G} \bullet d\mathbf{r} = \oint_C \mathbf{G} \bullet \hat{\mathbf{T}}\, ds = \oint_C \mathbf{F} \bullet \hat{\mathbf{N}}\, ds.\end{aligned}$$

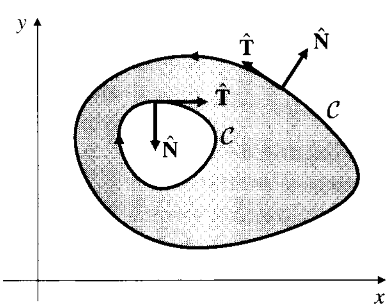

Figure 16.9 $\hat{\mathbf{N}} = \hat{\mathbf{T}} \times \mathbf{k}$

Exercises 16.3

1. Evaluate $\oint_C (\sin x + 3y^2)\,dx + (2x - e^{-y^2})\,dy$, where C is the boundary of the half-disk $x^2 + y^2 \le a^2$, $y \ge 0$, oriented counterclockwise.

2. Evaluate $\oint_C (x^2 - xy)\,dx + (xy - y^2)\,dy$ clockwise around the triangle with vertices $(0,0)$, $(1,1)$, and $(2,0)$.

3. Evaluate $\oint_C \left(x\sin(y^2) - y^2\right) dx + \left(x^2 y\cos(y^2) + 3x\right) dy$, where C is the counterclockwise boundary of the trapezoid with vertices $(0, -2)$, $(1, -1)$, $(1, 1)$, and $(0, 2)$.

4. Evaluate $\oint_C x^2 y\, dx - xy^2\, dy$, where C is the clockwise boundary of the region $0 \le y \le \sqrt{9 - x^2}$.

5. Use a line integral to find the plane area enclosed by the curve $\mathbf{r} = a\cos^3 t\,\mathbf{i} + b\sin^3 t\,\mathbf{j}$, $0 \le t \le 2\pi$.

6. We deduced the two-dimensional Divergence Theorem from Green's Theorem. Reverse the argument and use the two-dimensional Divergence Theorem to prove Green's Theorem.

7. Sketch the plane curve C: $\mathbf{r} = \sin t\,\mathbf{i} + \sin 2t\,\mathbf{j}$, $(0 \le t \le 2\pi)$. Evaluate $\oint_C \mathbf{F} \bullet d\mathbf{r}$, where $\mathbf{F} = ye^{x^2}\mathbf{i} + x^3 e^y \mathbf{j}$.

8. If C is the positively oriented boundary of a plane region R having area A and centroid $(\bar{x}, \bar{y})$, interpret geometrically the line integral $\oint_C \mathbf{F} \bullet d\mathbf{r}$, where (a) $\mathbf{F} = x^2\mathbf{j}$, (b) $\mathbf{F} = xy\mathbf{i}$, and (c) $\mathbf{F} = y^2\mathbf{i} + 3xy\mathbf{j}$.

* 9. **(Average values of harmonic functions)** If $u(x, y)$ is harmonic in a domain containing a disk of radius r with boundary C_r, then the average value of u around the circle is the value of u at the centre. Prove this by showing that the derivative of the average value with respect to r is zero (use divergence theorem and harmonicity of u) and the fact that the limit of the average value as $r \to 0$ is the value of u at the centre.

16.4 The Divergence Theorem in 3-Space

The **Divergence Theorem** (also called **Gauss's Theorem**) is one of two important versions of the Fundamental Theorem of Calculus in $\mathbb{R}^3$. (The other is Stokes's Theorem, presented in the next section.)

In the Divergence Theorem, the integral of the *derivative* $\mathbf{div}\,\mathbf{F} = \nabla \bullet \mathbf{F}$ over a domain in 3-space is expressed as the flux of $\mathbf{F}$ out of the surface of that domain. It therefore closely resembles the two-dimensional version Theorem 7 given in the previous section. The theorem holds for a general class of domains in $\mathbb{R}^3$ that are bounded by piecewise smooth closed surfaces. However, we will restrict our statement and proof of the theorem to domains of a special type. Extending the concept of an x-simple plane domain defined in Section 14.2, we say the three-dimensional domain D is **x-simple** if it is bounded by a piecewise smooth surface $\mathcal{S}$ and if every straight line parallel to the x-axis and passing through an interior point of D meets $\mathcal{S}$ at exactly two points. Similar definitions hold for y-simple and z-simple, and we call the domain D **regular** if it is a union of finitely many, nonoverlapping subdomains, each of which is x-simple, y-simple, and z-simple.

THEOREM 8

The Divergence Theorem (Gauss's Theorem)

Let D be a regular, 3-dimensional domain whose boundary $\mathcal{S}$ is an oriented, closed surface with unit normal field $\hat{\mathbf{N}}$ pointing out of D. If $\mathbf{F}$ is a smooth vector field defined on D, then

$$\iiint_D \mathbf{div}\,\mathbf{F}\, dV = \oiint_{\mathcal{S}} \mathbf{F} \bullet \hat{\mathbf{N}}\, dS.$$

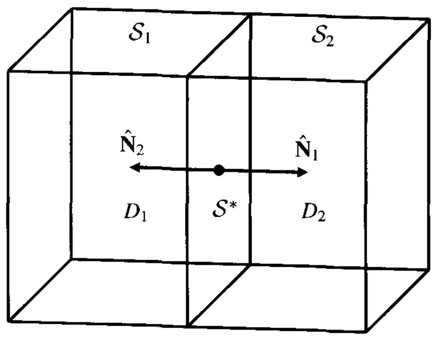

Figure 16.10 A union of abutting domains

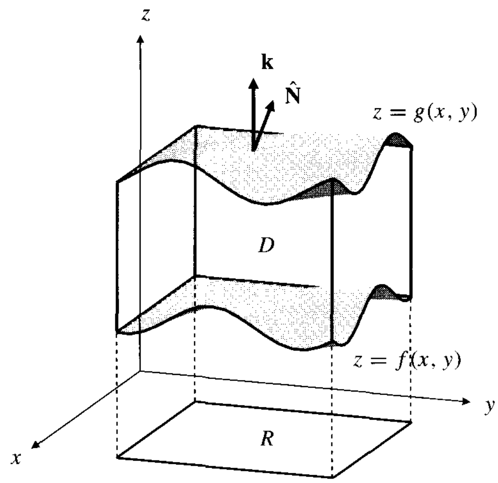

Figure 16.11 A z-simple domain

PROOF Since the domain D is a union of finitely many nonoverlapping domains that are x-simple, y-simple, and z-simple, it is sufficient to prove the theorem for a subdomain of D with this property. To see this, suppose, for instance, that D and $\mathcal{S}$ are each divided into two parts, D_1 and D_2, and $\mathcal{S}_1$ and $\mathcal{S}_2$, by a surface $\mathcal{S}^*$ slicing through D. (See Figure 16.10.) $\mathcal{S}^*$ is part of the boundary of both D_1 and D_2, but the exterior normals, $\hat{\mathbf{N}}_1$ and $\hat{\mathbf{N}}_2$, of the two subdomains point in opposite directions on either side of $\mathcal{S}^*$. If the formula in the theorem holds for both subdomains,

$$\iiint_{D_1} \mathbf{div}\,\mathbf{F}\,dV = \oiint_{\mathcal{S}_1 \cup \mathcal{S}^*} \mathbf{F} \bullet \hat{\mathbf{N}}_1\,dS$$
$$\iiint_{D_2} \mathbf{div}\,\mathbf{F}\,dV = \oiint_{\mathcal{S}_2 \cup \mathcal{S}^*} \mathbf{F} \bullet \hat{\mathbf{N}}_2\,dS,$$

then, adding these equations, we get

$$\iiint_{D} \mathbf{div}\,\mathbf{F}\,dV = \oiint_{\mathcal{S}_1 \cup \mathcal{S}_2} \mathbf{F} \bullet \hat{\mathbf{N}}\,dS = \oiint_{\mathcal{S}} \mathbf{F} \bullet \hat{\mathbf{N}}\,dS;$$

the contributions from $\mathcal{S}^*$ cancel out because on that surface $\hat{\mathbf{N}}_2 = -\hat{\mathbf{N}}_1$.

For the rest of this proof we assume, therefore, that D is x-, y-, and z-simple. Since D is z-simple, it lies between the graphs of two functions defined on a region R in the xy-plane; if (x, y, z) is in D, then (x, y) is in R and $f(x, y) \le z \le g(x, y)$. (See Figure 16.11.) We have

$$\begin{aligned}\iiint_D \frac{\partial F_3}{\partial z}\,dV &= \iint_R dx\,dy \int_{f(x,y)}^{g(x,y)} \frac{\partial F_3}{\partial z}\,dz \\ &= \iint_R \Big(F_3\big(x, y, g(x, y)\big) - F_3\big(x, y, f(x, y)\big)\Big)\,dx\,dy.\end{aligned}$$

Now

$$\oiint_{\mathcal{S}} \mathbf{F} \bullet \hat{\mathbf{N}}\,dS = \oiint_{\mathcal{S}} \Big(F_1\,\mathbf{i} \bullet \hat{\mathbf{N}} + F_2\,\mathbf{j} \bullet \hat{\mathbf{N}} + F_3\,\mathbf{k} \bullet \hat{\mathbf{N}}\Big)\,dS.$$

Only the last term involves F_3, and it can be split into three integrals, over the top surface $z = g(x, y)$, the bottom surface $z = f(x, y)$, and vertical side wall lying above the boundary of R:

$$\oiint_S F_3(x, y, z)\,\mathbf{k} \bullet \hat{\mathbf{N}}\,dS = \left(\iint_{\text{top}} + \iint_{\text{bottom}} + \iint_{\text{side}}\right) F_3(x, y, z)\,\mathbf{k} \bullet \hat{\mathbf{N}}\,dS.$$

On the side wall, $\mathbf{k} \bullet \hat{\mathbf{N}} = 0$, so that integral is zero. On the top surface, $z = g(x, y)$, and the vector area element is

$$\hat{\mathbf{N}}\,dS = \left(-\frac{\partial g}{\partial x}\mathbf{i} - \frac{\partial g}{\partial y}\mathbf{j} + \mathbf{k}\right) dx\,dy.$$

Accordingly,

$$\iint_{\text{top}} F_3(x, y, z)\,\mathbf{k} \bullet \hat{\mathbf{N}}\,dS = \iint_R F_3\big(x, y, g(x, y)\big)\,dx\,dy.$$

Similarly, we have

$$\iint_{\text{bottom}} F_3(x, y, z)\,\mathbf{k} \bullet \hat{\mathbf{N}}\,dS = -\iint_R F_3\big(x, y, f(x, y)\big)\,dx\,dy;$$

the negative sign occurs because $\hat{\mathbf{N}}$ points down rather than up on the bottom. Thus we have shown that

$$\iiint_D \frac{\partial F_3}{\partial z}\,dV = \oiint_S F_3\,\mathbf{k} \bullet \hat{\mathbf{N}}\,dS.$$

Similarly, because D is also x-simple and y-simple,

$$\iiint_D \frac{\partial F_1}{\partial x}\,dV = \oiint_S F_1\,\mathbf{i} \bullet \hat{\mathbf{N}}\,dS$$

$$\iiint_D \frac{\partial F_2}{\partial y}\,dV = \oiint_S F_2\,\mathbf{j} \bullet \hat{\mathbf{N}}\,dS.$$

Adding these three results we get

$$\iiint_D \mathbf{div}\,\mathbf{F}\,dV = \oiint_S \mathbf{F} \bullet \hat{\mathbf{N}}\,dS.$$

The Divergence Theorem can be used in both directions to simplify explicit calculations of surface integrals or volumes. We give examples of each.

Example 1 Let $\mathbf{F} = bxy^2\mathbf{i} + bx^2y\mathbf{j} + (x^2 + y^2)z^2\mathbf{k}$, and let $\mathcal{S}$ be the closed surface bounding the solid cylinder R defined by $x^2 + y^2 \le a^2$ and $0 \le z \le b$. Find $\oiint_S \mathbf{F} \bullet d\mathbf{S}$.

Solution By the Divergence Theorem,

$$\begin{aligned}\oiint_S \mathbf{F} \bullet d\mathbf{S} &= \iiint_R \mathbf{div}\,\mathbf{F}\,dV = \iiint_R (x^2 + y^2)(b + 2z)\,dV \\ &= \int_0^b (b + 2z)\,dz \int_0^{2\pi} d\theta \int_0^a r^2\,r\,dr \\ &= (b^2 + b^2)2\pi(a^4/4) = \pi a^4 b^2.\end{aligned}$$

Example 2 Evaluate $\oint\!\!\oint_S (x^2+y^2)\,dS$, where S is the sphere $x^2+y^2+z^2=a^2$. Use the Divergence Theorem.

Solution On S we have

$$\hat{\mathbf{N}} = \frac{\mathbf{r}}{a} = \frac{x\mathbf{i}+y\mathbf{j}+z\mathbf{k}}{a}.$$

We would like to choose $\mathbf{F}$ so that $\mathbf{F} \bullet \hat{\mathbf{N}} = x^2+y^2$. Observe that $\mathbf{F} = a(x\mathbf{i}+y\mathbf{j})$ will do. If B is the ball bounded by S, then

$$\begin{aligned}\oint\!\!\oint_S (x^2+y^2)\,dS &= \oint\!\!\oint_S \mathbf{F} \bullet \hat{\mathbf{N}}\,dS = \iiint_B \mathbf{div}\,\mathbf{F}\,dV \\ &= \iiint_B 2a\,dV = (2a)\frac{4}{3}\pi a^3 = \frac{8}{3}\pi a^4.\end{aligned}$$

■

Example 3 By using the Divergence Theorem with $\mathbf{F} = x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$, calculate the volume of a cone having base area A and height h. The base can be any smoothly bounded plane region.

Solution Let the vertex of the cone be at the origin and the base in the plane $z=h$ as shown in Figure 16.12. The solid cone C has surface consisting of two parts: the conical wall S and the base region D that has area A. Since $\mathbf{F}(x,y,z)$ points directly away from the origin at any point $(x,y,z) \neq (0,0,0)$, we have $\mathbf{F} \bullet \hat{\mathbf{N}} = 0$ on S. On D, we have $\hat{\mathbf{N}} = \mathbf{k}$ and $z=h$, so $\mathbf{F} \bullet \hat{\mathbf{N}} = z = h$ on the base of the cone. Since $\mathbf{div}\,\mathbf{F}(x,y,z) = 1+1+1 = 3$, we have, by the Divergence Theorem,

$$\begin{aligned}3V = \iiint_C \mathbf{div}\,\mathbf{F}\,dV &= \iint_S \mathbf{F} \bullet \hat{\mathbf{N}}\,dS + \iint_D \mathbf{F} \bullet \hat{\mathbf{N}}\,dS \\ &= 0 + h\iint_D dS = Ah.\end{aligned}$$

Thus $V = \frac{1}{3}Ah$, the well-known formula for the volume of a cone.

■

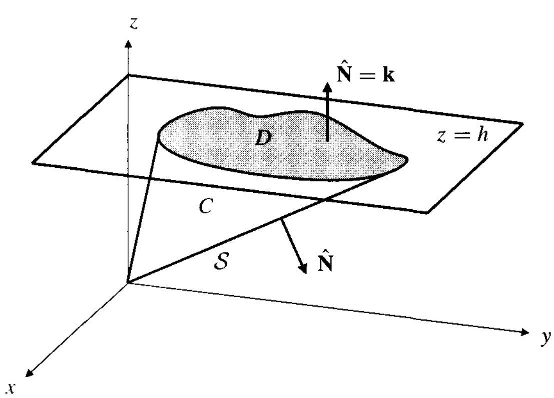

Figure 16.12 A cone with an arbitrarily shaped base

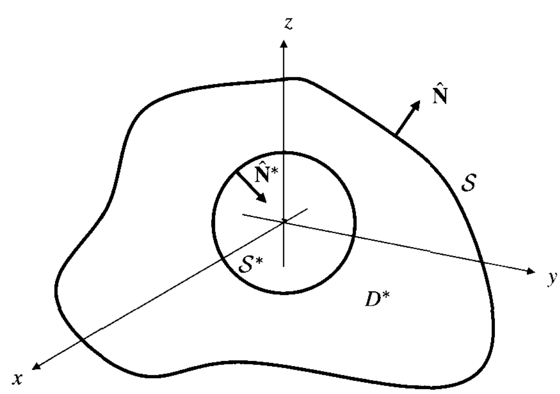

Figure 16.13 A solid domain with a spherical cavity

Example 4 Let $\mathcal{S}$ be the surface of an arbitrary regular domain D in 3-space that contains the origin in its interior. Find

$$\oiint_{\mathcal{S}} \mathbf{F} \bullet \hat{\mathbf{N}}\, dS,$$

where $\mathbf{F}(\mathbf{r}) = m\mathbf{r}/|\mathbf{r}|^3$ and $\hat{\mathbf{N}}$ is the unit outward normal on $\mathcal{S}$.

Solution Since $\mathbf{F}$ and, therefore, $\mathbf{div}\,\mathbf{F}$ are undefined at the origin, we cannot apply the Divergence Theorem directly. To overcome this problem we use a little trick. Let $\mathcal{S}^*$ be a small sphere centred at the origin bounding a ball contained wholly in D. (See Figure 16.13.) Let $\hat{\mathbf{N}}^*$ be the unit normal on $\mathcal{S}^*$ pointing *into* the sphere, and let D^* be that part of D that lies outside $\mathcal{S}^*$. As shown in Example 3 of Section 16.1, $\mathbf{div}\,\mathbf{F} = 0$ on D^*. Also,

$$\oiint_{\mathcal{S}^*} \mathbf{F} \bullet \hat{\mathbf{N}}^*\, dS = -4\pi m,$$

is the flux of $\mathbf{F}$ *inward* through the sphere $\mathcal{S}^*$. (See Example 1 of Section 15.6.) Therefore,

$$\begin{aligned} 0 = \iiint_{D^*} \mathbf{div}\,\mathbf{F}\, dV &= \oiint_{\mathcal{S}} \mathbf{F} \bullet \hat{\mathbf{N}}\, dS + \oiint_{\mathcal{S}^*} \mathbf{F} \bullet \hat{\mathbf{N}}^*\, dS \\ &= \oiint_{\mathcal{S}} \mathbf{F} \bullet \hat{\mathbf{N}}\, dS - 4\pi m, \end{aligned}$$

so $\displaystyle\oiint_{\mathcal{S}} \mathbf{F} \bullet \hat{\mathbf{N}}\, dS = 4\pi m.$

Example 5 Find the flux of $\mathbf{F} = x\mathbf{i} + y^2\mathbf{j} + z\mathbf{k}$ upward through the first-octant part $\mathcal{S}$ of the cylindrical surface $x^2 + z^2 = a^2$, $0 \le y \le b$.

Solution $\mathcal{S}$ is one of five surfaces that form the boundary of the solid region D shown in Figure 16.14. The other four surfaces are planar: $\mathcal{S}_1$ lies in the plane $z = 0$, $\mathcal{S}_2$ lies in the plane $x = 0$, $\mathcal{S}_3$ lies in the plane $y = 0$, and $\mathcal{S}_4$ lies in the plane $y = b$. Orient all these surfaces with normal $\hat{\mathbf{N}}$ pointing out of D. On $\mathcal{S}_1$ we have $\hat{\mathbf{N}} = -\mathbf{k}$, so $\mathbf{F} \bullet \hat{\mathbf{N}} = -z = 0$ on $\mathcal{S}_1$. Similarly, $\mathbf{F} \bullet \hat{\mathbf{N}} = 0$ on $\mathcal{S}_2$ and $\mathcal{S}_3$. On $\mathcal{S}_4$, $y = b$ and $\hat{\mathbf{N}} = \mathbf{j}$, so $\mathbf{F} \bullet \hat{\mathbf{N}} = y^2 = b^2$ there. If $\mathcal{S}_{\text{tot}}$ denotes the whole boundary of D, then

$$\begin{aligned} \oiint_{\mathcal{S}_{\text{tot}}} \mathbf{F} \bullet \hat{\mathbf{N}}\, dS &= \iint_{\mathcal{S}} \mathbf{F} \bullet \hat{\mathbf{N}}\, dS + 0 + 0 + 0 + \iint_{\mathcal{S}_4} \mathbf{F} \bullet \hat{\mathbf{N}}\, dS \\ &= \iint_{\mathcal{S}} \mathbf{F} \bullet \hat{\mathbf{N}}\, dS + \frac{\pi a^2 b^2}{4}. \end{aligned}$$

On the other hand, by the Divergence Theorem,

$$\oiint_{\mathcal{S}_{\text{tot}}} \mathbf{F} \bullet \hat{\mathbf{N}}\, dS = \iiint_D \mathbf{div}\,\mathbf{F}\, dV = \iiint_D (2 + 2y)\, dV = 2V + 2V\bar{y},$$

where $V = \pi a^2 b/4$ is the volume of D, and $\bar{y} = b/2$ is the y-coordinate of the centroid of D. Combining these results, the flux of $\mathbf{F}$ upward through $\mathcal{S}$ is

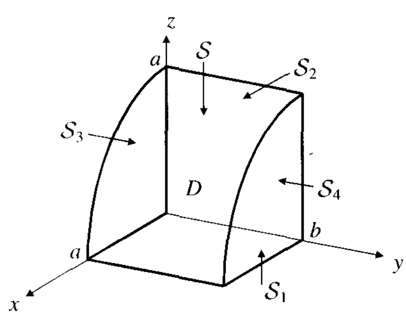

Figure 16.14 The boundary of domain D has five faces, one curved and four planar

$$\oiint_S \mathbf{F} \bullet \hat{\mathbf{N}}\, dS = \frac{2\pi a^2 b}{4}\left(1 + \frac{b}{2}\right) - \frac{\pi a^2 b^2}{4} = \frac{\pi a^2 b}{2}.$$

Among the examples above, Example 4 is the most significant and the one that best represents the way that the Divergence Theorem is used in practice. It is predominantly a theoretical tool, rather than a tool for calculation. We will look at some applications in Section 16.6.

Variants of the Divergence Theorem

Other versions of the Fundamental Theorem of Calculus can be derived from the Divergence Theorem. Two are given in the following theorem:

THEOREM 9

If D satisfies the conditions of the Divergence Theorem and has surface $\mathcal{S}$, and if $\mathbf{F}$ is a smooth vector field and ϕ is a smooth scalar field, then

(a) $$\iiint_D \mathbf{curl}\,\mathbf{F}\, dV = -\oiint_S \mathbf{F} \times \hat{\mathbf{N}}\, dS,$$

(b) $$\iiint_D \mathbf{grad}\,\phi\, dV = \oiint_S \phi \hat{\mathbf{N}}\, dS.$$

PROOF Observe that both of these formulas are equations of *vectors*. They are derived by applying the Divergence Theorem to $\mathbf{F} \times \mathbf{c}$ and $\phi\mathbf{c}$, respectively, where $\mathbf{c}$ is an arbitrary constant vector. We give the details for formula (a) and leave (b) as an exercise.

Using Theorem 3(d), we calculate

$$\nabla \bullet (\mathbf{F} \times \mathbf{c}) = (\nabla \times \mathbf{F}) \bullet \mathbf{c} - \mathbf{F} \bullet (\nabla \times \mathbf{c}) = (\nabla \times \mathbf{F}) \bullet \mathbf{c}.$$

Also, by the scalar triple product identity (see Exercise 18 of Section 10.3),

$$(\mathbf{F} \times \mathbf{c}) \bullet \hat{\mathbf{N}} = (\hat{\mathbf{N}} \times \mathbf{F}) \bullet \mathbf{c} = -(\mathbf{F} \times \hat{\mathbf{N}}) \bullet \mathbf{c}.$$

Therefore,

$$\begin{aligned}\left(\iiint_D \mathbf{curl}\,\mathbf{F}\, dV + \oiint_S \mathbf{F} \times \hat{\mathbf{N}}\, dS\right) \bullet \mathbf{c} \\ &= \iiint_D (\nabla \times \mathbf{F}) \bullet \mathbf{c}\, dV - \oiint_S (\mathbf{F} \times \mathbf{c}) \bullet \hat{\mathbf{N}}\, dS \\ &= \iiint_D \mathbf{div}\,(\mathbf{F} \times \mathbf{c})\, dV - \oiint_S (\mathbf{F} \times \mathbf{c}) \bullet \hat{\mathbf{N}}\, dS = 0.\end{aligned}$$

Since $\mathbf{c}$ is arbitrary, the vector in the large parentheses must be the zero vector. (If $\mathbf{c} \bullet \mathbf{a} = 0$ for every vector $\mathbf{c}$, then $\mathbf{a} = \mathbf{0}$.) This establishes formula (a).

Exercises 16.4

In Exercises 1–4, use the Divergence Theorem to calculate the flux of the given vector field out of the sphere $\mathcal{S}$ with equation $x^2 + y^2 + z^2 = a^2$, where $a > 0$.

1. $\mathbf{F} = x\mathbf{i} - 2y\mathbf{j} + 4z\mathbf{k}$

2. $\mathbf{F} = ye^z\mathbf{i} + x^2e^z\mathbf{j} + xy\mathbf{k}$

3. $\mathbf{F} = (x^2 + y^2)\mathbf{i} + (y^2 - z^2)\mathbf{j} + z\mathbf{k}$

4. $\mathbf{F} = x^3\mathbf{i} + 3yz^2\mathbf{j} + (3y^2z + x^2)\mathbf{k}$

In Exercises 5–8, evaluate the flux of $\mathbf{F} = x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}$ outward across the boundary of the given solid region.

5. The ball $(x-2)^2 + y^2 + (z-3)^2 \le 9$
6. The solid ellipsoid $x^2 + y^2 + 4(z-1)^2 \le 4$
7. The tetrahedron $x + y + z \le 3$, $x \ge 0$, $y \ge 0$, $z \ge 0$
8. The cylinder $x^2 + y^2 \le 2y$, $0 \le z \le 4$
9. Let A be the area of a region D forming part of the surface of a sphere of radius R centred at the origin, and let V be the volume of the solid cone C consisting of all points on line segments joining the centre of the sphere to points in D. Show that

$$V = \frac{1}{3}AR$$

by applying the Divergence Theorem to $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.

10. Let $\phi(x, y, z) = xy + z^2$. Find the flux of $\nabla\phi$ upward through the triangular planar surface $\mathcal{S}$ with vertices at $(a, 0, 0)$, $(0, b, 0)$, and $(0, 0, c)$.
11. A conical domain with vertex $(0, 0, b)$ and axis along the z-axis has as base a disk of radius a in the xy-plane. Find the flux of

$$\mathbf{F} = (x + y^2)\mathbf{i} + (3x^2y + y^3 - x^3)\mathbf{j} + (z + 1)\mathbf{k}$$

upward through the conical part of the surface of the domain.

12. Find the flux of $\mathbf{F} = (y + xz)\mathbf{i} + (y + yz)\mathbf{j} - (2x + z^2)\mathbf{k}$ upward through the first octant part of the sphere $x^2 + y^2 + z^2 = a^2$.
13. Let D be the region $x^2 + y^2 + z^2 \le 4a^2$, $x^2 + y^2 \ge a^2$. The surface $\mathcal{S}$ of D consists of a cylindrical part, $\mathcal{S}_1$, and a spherical part, $\mathcal{S}_2$. Evaluate the flux of

$$\mathbf{F} = (x + yz)\mathbf{i} + (y - xz)\mathbf{j} + (z - e^x \sin y)\mathbf{k}$$

out of D through (a) the whole surface $\mathcal{S}$, (b) the surface $\mathcal{S}_1$, and (c) the surface $\mathcal{S}_2$.

14. Evaluate $\iint_{\mathcal{S}} (3xz^2\mathbf{i} - x\mathbf{j} - y\mathbf{k}) \bullet \hat{\mathbf{N}}\,dS$, where $\mathcal{S}$ is that part of the cylinder $y^2 + z^2 = 1$ which lies in the first octant and between the planes $x = 0$ and $x = 1$.
15. A solid region R has volume V and centroid at the point $(\bar{x}, \bar{y}, \bar{z})$. Find the flux of

$$\mathbf{F} = (x^2 - x - 2y)\mathbf{i} + (2y^2 + 3y - z)\mathbf{j} - (z^2 - 4z + xy)\mathbf{k}$$

out of R through its surface.

16. The plane $x + y + z = 0$ divides the cube $-1 \le x \le 1$, $-1 \le y \le 1$, $-1 \le z \le 1$ into two parts. Let the lower part (with one vertex at $(-1, -1, -1)$) be D. Sketch D. Note that it has seven faces, one of which is hexagonal. Find the flux of $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ out of D through each of its faces.
17. Let $\mathbf{F} = (x^2 + y + 2 + z^2)\mathbf{i} + (e^{x^2} + y^2)\mathbf{j} + (3 + x)\mathbf{k}$. Let $a > 0$, and let $\mathcal{S}$ be the part of the spherical surface $x^2 + y^2 + z^2 = 2az + 3a^2$ that is above the xy-plane. Find the flux of $\mathbf{F}$ outward across $\mathcal{S}$.
18. A pile of wet sand having total volume 5π covers the disk $x^2 + y^2 \le 1$, $z = 0$. The momentum of water vapour is given by $\mathbf{F} = \mathbf{grad}\,\phi + \mu\,\mathbf{curl}\,\mathbf{G}$, where $\phi = x^2 - y^2 + z^2$ is the water concentration, $\mathbf{G} = \frac{1}{3}(-y^3\mathbf{i} + x^3\mathbf{j} + z^3\mathbf{k})$, and μ is a constant. Find the flux of $\mathbf{F}$ upward through the top surface of the sand pile.

In Exercises 19–29, D is a three-dimensional domain satisfying the conditions of the Divergence Theorem, and $\mathcal{S}$ is its surface. $\hat{\mathbf{N}}$ is the unit outward (from D) normal field on $\mathcal{S}$. The functions ϕ and ψ are smooth scalar fields on D. Also, $\partial\phi/\partial n$ denotes the first directional derivative of ϕ in the direction of $\hat{\mathbf{N}}$ at any point on $\mathcal{S}$:

$$\frac{\partial\phi}{\partial n} = \nabla\phi \bullet \hat{\mathbf{N}}.$$

19. Show that $\oint\!\!\oint_{\mathcal{S}} \mathbf{curl}\,\mathbf{F} \bullet \hat{\mathbf{N}}\,dS = 0$, where $\mathbf{F}$ is an arbitrary smooth vector field.
20. Show that the volume V of D is given by

$$V = \frac{1}{3}\oint\!\!\oint_{\mathcal{S}} (x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) \bullet \hat{\mathbf{N}}\,dS.$$

21. If D has volume V, show that

$$\bar{\mathbf{r}} = \frac{1}{2V}\oint\!\!\oint_{\mathcal{S}} (x^2 + y^2 + z^2)\hat{\mathbf{N}}\,dS$$

is the position vector of the centre of gravity of D.

22. Show that $\oint\!\!\oint_{\mathcal{S}} \nabla\phi \times \hat{\mathbf{N}}\,dS = 0$.
23. If $\mathbf{F}$ is a smooth vector field on D, show that

$$\iiint_D \phi\,\mathbf{div}\,\mathbf{F}\,dV + \iiint_D \nabla\phi \bullet \mathbf{F}\,dV = \oint\!\!\oint_{\mathcal{S}} \phi\mathbf{F} \bullet \hat{\mathbf{N}}\,dS.$$

Hint: use Theorem 3(b) from Section 16.2.

Properties of the Laplacian operator

24. If $\nabla^2\phi = 0$ in D and $\phi(x, y, z) = 0$ on $\mathcal{S}$, show that $\phi(x, y, z) = 0$ in D. *Hint:* let $\mathbf{F} = \nabla\phi$ in Exercise 23.
25. **(Uniqueness for the Dirichlet problem)** The Dirichlet problem for the Laplacian operator is the boundary-value problem

$$\begin{cases} \nabla^2 u(x, y, z) = f(x, y, z) & \text{on } D \\ u(x, y, z) = g(x, y, z) & \text{on } \mathcal{S}, \end{cases}$$

where f and g are given functions defined on D and $\mathcal{S}$, respectively. Show that this problem can have at most one solution $u(x, y, z)$. *Hint:* suppose there are two solutions, u and v, and apply Exercise 24 to their difference $\phi = u - v$.

26. **(The Neumann problem)** If $\nabla^2\phi = 0$ in D and $\partial\phi/\partial n = 0$ on $\mathcal{S}$, show that $\nabla\phi(x, y, z) = 0$ on D. The Neumann problem for the Laplacian operator is the boundary-value problem

$$\begin{cases} \nabla^2 u(x, y, z) = f(x, y, z) & \text{on } D \\ \dfrac{\partial}{\partial n} u(x, y, z) = g(x, y, z) & \text{on } \mathcal{S}, \end{cases}$$

where f and g are given functions defined on D and $\mathcal{S}$, respectively. Show that, if D is connected, then any two solutions of the Neumann problem must differ by a constant on D.

27. Verify that $\displaystyle\iiint_D \nabla^2\phi\, dV = \oiint_{\mathcal{S}} \frac{\partial\phi}{\partial n}\, dS.$

28. Verify that

$$\iiint_D \left(\phi\nabla^2\psi - \psi\nabla^2\phi\right) dV = \oiint_{\mathcal{S}} \left(\phi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\phi}{\partial n}\right) dS.$$

29. By applying the Divergence Theorem to $\mathbf{F} = \phi\mathbf{c}$, where $\mathbf{c}$ is an arbitrary constant vector, show that

$$\iiint_D \nabla\phi\, dV = \oiint_{\mathcal{S}} \phi\hat{\mathbf{N}}\, dS.$$

***30.** Let P_0 be a fixed point, and for each $\epsilon > 0$ let D_ϵ be a domain with boundary $\mathcal{S}_\epsilon$ satisfying the conditions of the Divergence Theorem. Suppose that the maximum distance from P_0 to points P in D_ϵ approaches zero as $\epsilon \to 0^+$. If D_ϵ has volume $\mathrm{vol}(D_\epsilon)$, show that

$$\lim_{\epsilon\to 0^+} \frac{1}{\mathrm{vol}(D_\epsilon)} \oiint_{\mathcal{S}_\epsilon} \mathbf{F} \bullet \hat{\mathbf{N}}\, dS = \mathbf{div}\,\mathbf{F}(P_0).$$

This generalizes Theorem 1 of Section 16.1.

16.5 Stokes's Theorem

If we regard a region R in the xy-plane as a surface in 3-space with normal field $\hat{\mathbf{N}} = \mathbf{k}$, the Green's Theorem formula can be written in the form

$$\oint_C \mathbf{F} \bullet d\mathbf{r} = \iint_R \mathbf{curl}\,\mathbf{F} \bullet \hat{\mathbf{N}}\, dS.$$

Stokes's Theorem given below generalizes this to nonplanar surfaces.

THEOREM 10 **Stokes's Theorem**

Let $\mathcal{S}$ be a piecewise smooth, oriented surface in 3-space, having unit normal field $\hat{\mathbf{N}}$ and boundary $\mathcal{C}$ consisting of one or more piecewise smooth, closed curves with orientation inherited from $\mathcal{S}$. If $\mathbf{F}$ is a smooth vector field defined on an open set containing $\mathcal{S}$, then

$$\oint_{\mathcal{C}} \mathbf{F} \bullet d\mathbf{r} = \iint_{\mathcal{S}} \mathbf{curl}\,\mathbf{F} \bullet \hat{\mathbf{N}}\, dS.$$

PROOF An argument similar to those given in the proofs of Green's Theorem and the Divergence Theorem shows that if $\mathcal{S}$ is decomposed into finitely many nonoverlapping subsurfaces, then it is sufficient to prove that the formula above holds for each of them. (If subsurfaces $\mathcal{S}_1$ and $\mathcal{S}_2$ meet along the curve $\mathcal{C}^*$, then $\mathcal{C}^*$ inherits opposite orientations as part of the boundaries of $\mathcal{S}_1$ and $\mathcal{S}_2$, so the line integrals along $\mathcal{C}^*$ cancel out. See Figure 16.15(a).) We can subdivide $\mathcal{S}$ into enough smooth subsurfaces that each one has a one-to-one normal projection onto a coordinate plane. We will establish the formula for one such subsurface, which we will now call $\mathcal{S}$.

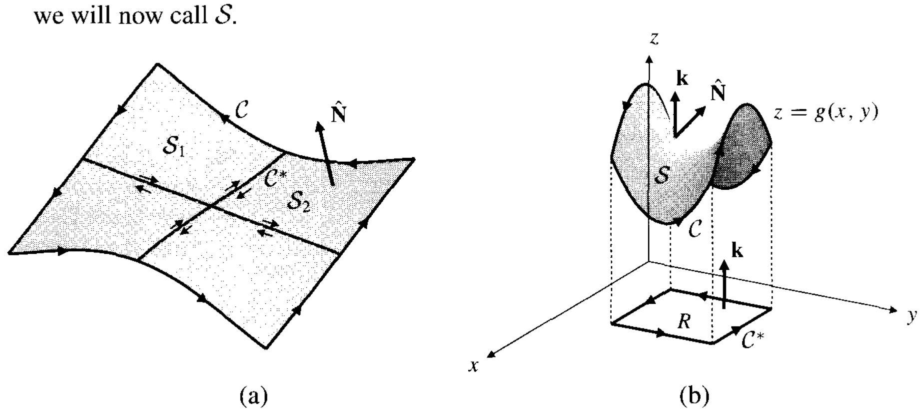

Figure 16.15

(a) Stokes's Theorem holds for a composite surface comprised of non-overlapping subsurfaces for which it is true

(b) A surface with a one-to-one projection on the xy-plane

Without loss of generality, assume that $\mathcal{S}$ has a one-to-one normal projection onto the xy-plane and that its normal field $\hat{\mathbf{N}}$ points upward. Therefore, on $\mathcal{S}$, z is a smooth function of x and y, say $z = g(x, y)$, defined for (x, y) in a region R of the xy-plane. The boundaries $\mathcal{C}$ of $\mathcal{S}$and $\mathcal{C}^*$ of R are both oriented counterclockwise as seen from a point high on the z-axis. (See Figure 16.15(b).) The normal field on $\mathcal{S}$ is

$$\hat{\mathbf{N}} = \frac{-\dfrac{\partial g}{\partial x}\mathbf{i} - \dfrac{\partial g}{\partial y}\mathbf{j} + \mathbf{k}}{\sqrt{1 + \left(\dfrac{\partial g}{\partial x}\right)^2 + \left(\dfrac{\partial g}{\partial y}\right)^2}},$$

and the surface area element on $\mathcal{S}$ is expressed in terms of the area element $dA = dx\,dy$ in the xy-plane as

$$dS = \sqrt{1 + \left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2}\,dA.$$

Therefore,

$$\iint_{\mathcal{S}} \mathbf{curl}\,\mathbf{F} \bullet \hat{\mathbf{N}}\,dS = \iint_R \left[\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\left(-\frac{\partial g}{\partial x}\right) + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\left(-\frac{\partial g}{\partial y}\right) + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\right] dA.$$

Since $z = g(x, y)$ on $\mathcal{C}$, we have $dz = \dfrac{\partial g}{\partial x}\,dx + \dfrac{\partial g}{\partial y}\,dy$. Thus,

$$\begin{aligned}\oint_{\mathcal{C}} \mathbf{F} \bullet d\mathbf{r} &= \oint_{\mathcal{C}^*} \Big[F_1(x, y, z)\,dx + F_2(x, y, z)\,dy \\ &\qquad + F_3(x, y, z)\left(\frac{\partial g}{\partial x}\,dx + \frac{\partial g}{\partial y}\,dy\right)\Big] \\ &= \oint_{\mathcal{C}^*} \left(\left[F_1(x, y, z) + F_3(x, y, z)\frac{\partial g}{\partial x}\right] dx \right. \\ &\qquad \left. + \left[F_2(x, y, z) + F_3(x, y, z)\frac{\partial g}{\partial y}\right] dy\right).\end{aligned}$$

We now apply Green's Theorem in the xy-plane to obtain

$$\begin{aligned}\oint_C \mathbf{F} \bullet d\mathbf{r} &= \iint_R \left(\frac{\partial}{\partial x} \left[F_2(x,y,z) + F_3(x,y,z) \frac{\partial g}{\partial y} \right] \right. \\ &\qquad \left. - \frac{\partial}{\partial y} \left[F_1(x,y,z) + F_3(x,y,z) \frac{\partial g}{\partial x} \right] \right) dA \\ &= \iint_R \left(\frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial z}\frac{\partial g}{\partial x} + \frac{\partial F_3}{\partial x}\frac{\partial g}{\partial y} + \frac{\partial F_3}{\partial z}\frac{\partial g}{\partial x}\frac{\partial g}{\partial y} + F_3 \frac{\partial^2 g}{\partial x \partial y} \right. \\ &\qquad \left. - \frac{\partial F_1}{\partial y} - \frac{\partial F_1}{\partial z}\frac{\partial g}{\partial y} - \frac{\partial F_3}{\partial y}\frac{\partial g}{\partial x} - \frac{\partial F_3}{\partial z}\frac{\partial g}{\partial y}\frac{\partial g}{\partial x} - F_3 \frac{\partial^2 g}{\partial y \partial x} \right) dA.\end{aligned}$$

Observe that four terms in the final integrand cancel out, and the remaining terms are equal to the terms in the expression for $\iint_S \mathbf{curl}\,\mathbf{F} \bullet \hat{\mathbf{N}}\, dS$ calculated above. Therefore the proof is complete.

Remark If $\mathbf{curl}\,\mathbf{F} = \mathbf{0}$ on a domain D with the property that every piecewise smooth, non-self-intersecting, closed curve in D is the boundary of a piecewise smooth surface in D, then Stokes's Theorem assures us that $\oint_C \mathbf{F} \bullet d\mathbf{r} = 0$ for *every* such curve $\mathcal{C}$; therefore $\mathbf{F}$ must be conservative. A simply connected domain D does have the property specified above. We will not attempt a formal proof of this topological fact here, but it should seem plausible if you recall the definition of simple connectedness. A closed curve $\mathcal{C}$ in a simply connected domain D must be able to shrink to a point in D without ever passing out of D. In so shrinking, it traces out a surface in D. This is why Theorem 4 of Section 16.2 is valid for simply connected domains.

Example 1 Evaluate $\oint_C \mathbf{F} \bullet d\mathbf{r}$, where $\mathbf{F} = -y^3\mathbf{i} + x^3\mathbf{j} - z^3\mathbf{k}$, and $\mathcal{C}$ is the curve of intersection of the cylinder $x^2 + y^2 = 1$ and the plane $2x + 2y + z = 3$ oriented so as to have a counterclockwise projection onto the xy-plane.

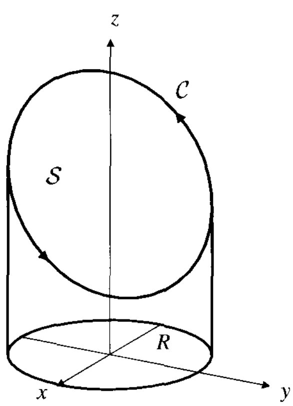

Figure 16.16

Solution $\mathcal{C}$ is the oriented boundary of an elliptic disk $\mathcal{S}$ that lies in the plane $2x + 2y + z = 3$ and has the circular disk R: $x^2 + y^2 \le 1$ as projection onto the xy-plane. (See Figure 16.16.) On $\mathcal{S}$ we have

$$\hat{\mathbf{N}}\, dS = (2\mathbf{i} + 2\mathbf{j} + \mathbf{k})\, dx\, dy.$$

Also,

$$\mathbf{curl}\,\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ -y^3 & x^3 & -z^3 \end{vmatrix} = 3(x^2 + y^2)\mathbf{k}.$$

Thus, by Stokes's Theorem,

$$\begin{aligned}\oint_C \mathbf{F} \bullet d\mathbf{r} &= \iint_S \mathbf{curl}\,\mathbf{F} \bullet \hat{\mathbf{N}}\, dS \\ &= \iint_R 3(x^2 + y^2)\, dx\, dy = 2\pi \int_0^1 3r^2\, r\, dr = \frac{3\pi}{2}.\end{aligned}$$

■

As with the Divergence Theorem, the principal importance of Stokes's Theorem is as a theoretical tool. However, it can also simplify the calculation of circulation integrals such as the one in the previous example. It is not difficult to imagine integrals whose evaluation would be impossibly difficult without the use of Stokes's Theorem or the Divergence Theorem. In the following example we use Stokes's Theorem twice, but the result could be obtained just as easily by using the Divergence Theorem.

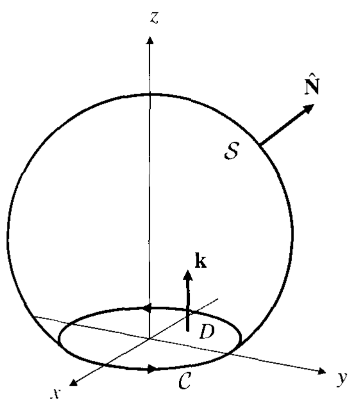

Figure 16.17 Part of a sphere and a disk with the same boundary

Example 2 Find $I = \iint_S \mathbf{curl\,F} \bullet \hat{\mathbf{N}}\,dS$, where $\mathcal{S}$ is that part of the sphere $x^2 + y^2 + (z-2)^2 = 8$ that lies above the xy-plane, $\hat{\mathbf{N}}$ is the unit outward normal field on $\mathcal{S}$, and

$$\mathbf{F} = y^2 \cos xz\,\mathbf{i} + x^3 e^{yz}\mathbf{j} - e^{-xyz}\mathbf{k}.$$

Solution The boundary, $\mathcal{C}$, of $\mathcal{S}$ is the circle $x^2 + y^2 = 4$ in the xy-plane, oriented counterclockwise as seen from the positive z-axis. (See Figure 16.17.) This curve is also the oriented boundary of the plane disk D: $x^2 + y^2 \le 4$, $z = 0$, with normal field $\hat{\mathbf{N}} = \mathbf{k}$. Thus, two applications of Stokes's Theorem give

$$I = \iint_{\mathcal{S}} \mathbf{curl\,F} \bullet \hat{\mathbf{N}}\,dS = \oint_{\mathcal{C}} \mathbf{F} \bullet d\mathbf{r} = \iint_D \mathbf{curl\,F} \bullet \mathbf{k}\,dA.$$

On D we have

$$\begin{aligned}\mathbf{curl\,F} \bullet \mathbf{k} &= \left(\frac{\partial}{\partial x}\left(x^3 e^{yz}\right) - \frac{\partial}{\partial y}\left(y^2 \cos xz\right)\right)\bigg|_{z=0} \\ &= 3x^2 - 2y.\end{aligned}$$

By symmetry, $\iint_D y\,dA = 0$, so

$$I = 3\iint_D x^2\,dA = 3\int_0^{2\pi} \cos^2\theta\,d\theta \int_0^2 r^3\,dr = 12\pi.$$

■

Remark A surface $\mathcal{S}$ satisfying the conditions of Stokes's Theorem may no longer do so if a single point is removed from it. An isolated boundary point of a surface is not an orientable curve, and Stokes's Theorem may therefore break down for such a surface. Consider, for example, the vector field

$$\mathbf{F} = \frac{\hat{\boldsymbol{\theta}}}{r} = -\frac{y}{x^2+y^2}\mathbf{i} + \frac{x}{x^2+y^2}\mathbf{j},$$

which is defined on the *punctured disk* D satisfying $0 < x^2 + y^2 \le a^2$. (See Example 3 in Section 16.3.) If D is oriented with upward normal $\mathbf{k}$, then its boundary consists of the oriented, smooth, closed curve, $\mathcal{C}$, given by $x = a\cos\theta$, $y = a\sin\theta$, $(0 \le \theta \le 2\pi)$, and the isolated point $(0, 0)$. We have

$$\begin{aligned}\oint_{\mathcal{C}} \mathbf{F} \bullet d\mathbf{r} &= \int_0^{2\pi} \left(\frac{-\sin\theta}{a}\mathbf{i} + \frac{\cos\theta}{a}\mathbf{j}\right) \bullet (-a\sin\theta\,\mathbf{i} + a\cos\theta\,\mathbf{j})\,d\theta \\ &= \int_0^{2\pi} (\sin^2\theta + \cos^2\theta)\,d\theta = 2\pi.\end{aligned}$$

However,

$$\mathbf{curl\,F} = \left[\frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) - \frac{\partial}{\partial y}\left(-\frac{y}{x^2+y^2}\right)\right]\mathbf{k} = \mathbf{0}$$

identically on D. Thus,

$$\iint_D \mathbf{curl\,F} \bullet \hat{\mathbf{N}}\,dS = 0,$$

and the conclusion of Stokes's Theorem fails in this case.

Exercises 16.5

1. Evaluate $\oint_C xy\,dx + yz\,dy + zx\,dz$ around the triangle with vertices $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$, oriented clockwise as seen from the point $(1, 1, 1)$.

2. Evaluate $\oint_C y\,dx - x\,dy + z^2\,dz$ around the curve C of intersection of the cylinders $z = y^2$ and $x^2 + y^2 = 4$, oriented counterclockwise as seen from a point high on the z-axis.

3. Evaluate $\iint_S \mathbf{curl\,F} \bullet \hat{\mathbf{N}}\,dS$, where S is the hemisphere $x^2 + y^2 + z^2 = a^2$, $z \ge 0$ with outward normal, and $\mathbf{F} = 3y\mathbf{i} - 2xz\mathbf{j} + (x^2 - y^2)\mathbf{k}$.

4. Evaluate $\iint_S \mathbf{curl\,F} \bullet \hat{\mathbf{N}}\,dS$, where S is the surface $x^2 + y^2 + 2(z-1)^2 = 6$, $z \ge 0$, $\hat{\mathbf{N}}$ is the unit outward (away from the origin) normal on S, and

$$\mathbf{F} = (xz - y^3\cos z)\mathbf{i} + x^3e^z\mathbf{j} + xyz\,e^{x^2+y^2+z^2}\mathbf{k}.$$

5. Use Stokes's Theorem to show that

$$\oint_C y\,dx + z\,dy + x\,dz = \sqrt{3}\pi a^2,$$

where C is the suitably oriented intersection of the surfaces $x^2 + y^2 + z^2 = a^2$ and $x + y + z = 0$.

6. Evaluate $\oint_C \mathbf{F} \bullet d\mathbf{r}$ around the curve

$$\mathbf{r} = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + \sin 2t\,\mathbf{k}, \quad (0 \le t \le 2\pi),$$

where

$$\mathbf{F} = (e^x - y^3)\mathbf{i} + (e^y + x^3)\mathbf{j} + e^z\mathbf{k}.$$

Hint: show that C lies on the surface $z = 2xy$.

7. Find the circulation of $\mathbf{F} = -y\mathbf{i} + x^2\mathbf{j} + z\mathbf{k}$ around the oriented boundary of the part of the paraboloid $z = 9 - x^2 - y^2$ lying above the xy-plane and having normal field pointing upward.

8. Evaluate $\oint_C \mathbf{F} \bullet d\mathbf{r}$, where

$$\mathbf{F} = ye^x\mathbf{i} + (x^2 + e^x)\mathbf{j} + z^2e^z\mathbf{k},$$

and C is the curve

$$\mathbf{r}(t) = (1 + \cos t)\mathbf{i} + (1 + \sin t)\mathbf{j} + (1 - \cos t - \sin t)\mathbf{k}$$

for $0 \le t \le 2\pi$. *Hint:* Use Stokes's Theorem, observing that C lies in a certain plane and has a circle as its projection onto the xy-plane. The integral can also be evaluated by using the techniques of Section 15.4.

9. Let C_1 be the straight line joining $(-1, 0, 0)$ to $(1, 0, 0)$, and let C_2 be the semicircle $x^2 + y^2 = 1$, $z = 0$, $y \ge 0$. Let S be a smooth surface joining C_1 to C_2 having upward normal, and let

$$\mathbf{F} = (\alpha x^2 - z)\mathbf{i} + (xy + y^3 + z)\mathbf{j} + \beta y^2(z+1)\mathbf{k}.$$

Find the values of α and β for which $I = \iint_S \mathbf{F} \bullet d\mathbf{S}$ is independent of the choice of S, and find the value of I for these values of α and β.

10. Let C be the curve $(x-1)^2 + 4y^2 = 16$, $2x + y + z = 3$, oriented counterclockwise when viewed from high on the z-axis. Let

$$\mathbf{F} = (z^2 + y^2 + \sin x^2)\mathbf{i} + (2xy + z)\mathbf{j}) + (xz + 2yz)\mathbf{k}.$$

Evaluate $\oint_C \mathbf{F} \bullet d\mathbf{r}$.

11. If $\mathcal{C}$ is the oriented boundary of surface $\mathcal{S}$, and ϕ and ψ are arbitrary smooth scalar fields, show that

$$\oint_{\mathcal{C}} \phi \nabla\psi \bullet d\mathbf{r} = -\oint_{\mathcal{C}} \psi \nabla\phi \bullet d\mathbf{r} = \iint_{\mathcal{S}} (\nabla\phi \times \nabla\psi) \bullet \hat{\mathbf{N}}\, dS.$$

Is $\nabla\phi \times \nabla\psi$ solenoidal? Find a vector potential for it.

12. Let $\mathcal{C}$ be a closed, non-self-intersecting, piecewise smooth plane curve in $\mathbb{R}^3$, which lies in a plane with unit normal $\hat{\mathbf{N}} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ and has orientation inherited from that of the plane. Show that the plane area enclosed by $\mathcal{C}$ is

$$\frac{1}{2}\oint_{\mathcal{C}} (bz - cy)\, dx + (cx - az)\, dy + (ay - bx)\, dz.$$

13. Use Stokes's Theorem to prove Theorem 2 of Section 16.1.

16.6 Some Physical Applications of Vector Calculus

In this section we will show how the theory developed in this chapter can be used to model concrete applied mathematical problems. We will look at two areas of application—fluid dynamics and electromagnetism—and will develop a few of the fundamental vector equations underlying these disciplines. Our purpose is to illustrate the techniques of vector calculus in applied contexts, rather than to provide any complete or even coherent introductions to the disciplines themselves.

Fluid Dynamics

Suppose that a region of 3-space is filled with a fluid (liquid or gas) in motion. Two approaches can be taken to describe the motion. We could attempt to determine the position, $\mathbf{r} = \mathbf{r}(a, b, c, t)$ at any time t, of a "particle" of fluid that was located at the point (a, b, c) at time $t = 0$. This is the Lagrange approach. Alternatively, we could attempt to determine the velocity, $\mathbf{v}(x, y, z, t)$, the density, $\delta(x, y, z, t)$, and other physical variables such as the pressure, $p(x, y, z, t)$, at any time t at any point (x, y, z) in the region occupied by the fluid. This is the Euler approach.

We will examine the latter method and describe how the Divergence Theorem can be used to translate some fundamental physical laws into equivalent mathematical equations. We assume throughout that the velocity, density, and pressure vary smoothly in all their variables and that the fluid is an *ideal fluid*, that is, nonviscous (it doesn't stick to itself), homogeneous, and isotropic (it has the same properties at all points and in all directions). Such properties are not always shared by real fluids, so we are dealing with a simplified mathematical model that does not always correspond exactly to the behaviour of real fluids.

Consider an imaginary closed surface $\mathcal{S}$ in the fluid, bounding a domain D. We call $\mathcal{S}$ "imaginary" because it is not a barrier that impedes the flow of the fluid in any way; it is just a means to concentrate our attention on a particular part of the fluid. It is fixed in space and does not move with the fluid. Let us assume that the fluid is being neither created nor destroyed anywhere (in particular, there are no sources or sinks), so the law of **conservation of mass** tells us that the rate of change of the mass of fluid in D equals the rate at which fluid enters D across $\mathcal{S}$.

The mass of fluid in volume element dV at position (x, y, z) at time t is $\delta(x, y, z, t)\, dV$, so the mass in D at time t is $\iiint_D \delta\, dV$. This mass changes at rate

$$\frac{\partial}{\partial t}\iiint_D \delta\, dV = \iiint_D \frac{\partial \delta}{\partial t}\, dV.$$

As we noted in Section 15.6, the volume of fluid passing *out* of D through area element dS at position (x, y, z) in the interval from time t to $t + dt$ is given by $\mathbf{v}(x, y, z, t) \bullet \hat{\mathbf{N}}\, dS\, dt$, where $\hat{\mathbf{N}}$ is the unit normal at (x, y, z) on $\mathcal{S}$ pointing out of D. Hence, the mass crossing dS outward in that time interval is $\delta\mathbf{v} \bullet \hat{\mathbf{N}}\, dS\, dt$, and the *rate* at which mass is flowing out of D across $\mathcal{S}$ at time t is

$$\oiint_{\mathcal{S}} \delta\mathbf{v} \bullet \hat{\mathbf{N}}\, dS.$$

The rate at which mass is flowing *into* D is the negative of the above rate. Since mass is conserved, we must have

$$\iiint_D \frac{\partial \delta}{\partial t}\, dV = -\oiint_{\mathcal{S}} \delta\mathbf{v} \bullet \hat{\mathbf{N}}\, dS = -\iiint_D \mathbf{div}\,(\delta\mathbf{v})\, dV,$$

where we have used the Divergence Theorem to replace the surface integral with a volume integral. Thus,

$$\iiint_D \left(\frac{\partial \delta}{\partial t} + \mathbf{div}\,(\delta\mathbf{v})\right) dV = 0.$$

This equation must hold for *any* domain D in the fluid.

If a continuous function f satisfies $\iiint_D f(P)\, dV = 0$ for every domain D, then $f(P) = 0$ at all points P, for if there were a point P_0 such that $f(P_0) \neq 0$ (say $f(P_0) > 0$), then, by continuity, f would be positive at all points in some sufficiently small ball B centred at P_0, and $\iiint_B f(P)\, dV$ would be greater than 0. Applying this principle, we must have

$$\frac{\partial \delta}{\partial t} + \mathbf{div}\,(\delta\mathbf{v}) = 0$$

throughout the fluid. This is called the **equation of continuity** for the fluid. It is equivalent to conservation of mass. Observe that if the fluid is **incompressible** then δ is a constant, independent of both time and spatial position. In this case $\partial\delta/\partial t = 0$ and $\mathbf{div}\,(\delta\mathbf{v}) = \delta\, \mathbf{div}\, \mathbf{v}$. Therefore, the equation of continuity for an incompressible fluid is simply

$$\mathbf{div}\, \mathbf{v} = 0.$$

The motion of the fluid is governed by Newton's Second Law, which asserts that the rate of change of momentum of any part of the fluid is equal to the sum of the forces applied to that part. Again, let us consider the part of the fluid in a domain D. At any time t its momentum is $\iiint_D \delta\mathbf{v}\, dV$ and is changing at rate

$$\iiint_D \frac{\partial}{\partial t}(\delta\mathbf{v})\, dV.$$

This change is due partly to momentum crossing $\mathcal{S}$ into or out of D (the momentum of the fluid crossing $\mathcal{S}$), partly to the pressure exerted on the fluid in D by the fluid outside, and partly to any external *body forces* (such as gravity or electromagnetic forces) acting on the fluid. Let us examine each of these causes in turn.

Momentum is transferred across $\mathcal{S}$ into D at the rate

$$-\oiint_S \mathbf{v}(\delta\mathbf{v} \bullet \hat{\mathbf{N}})\,dS.$$

The pressure on the fluid in D is exerted across $\mathcal{S}$ in the direction of the inward normal $-\hat{\mathbf{N}}$. Thus, this part of the force on the fluid in D is

$$-\oiint_S p\hat{\mathbf{N}}\,dS.$$

The body forces are best expressed in terms of the *force density* (force per unit mass), $\mathbf{F}$. The total body force on the fluid in D is therefore

$$\iiint_D \delta\mathbf{F}\,dV.$$

Newton's Second Law now implies that

$$\iiint_D \frac{\partial}{\partial t}(\delta\mathbf{v})\,dV = -\oiint_S \mathbf{v}(\delta\mathbf{v} \bullet \hat{\mathbf{N}})\,dS - \oiint_S p\hat{\mathbf{N}}\,dS + \iiint_D \delta\mathbf{F}\,dV.$$

Again, we would like to convert the surface integrals to triple integrals over D. If we use the results of Exercise 29 of Section 16.4 and Exercise 2 below, we get

$$\oiint_S p\hat{\mathbf{N}}\,dS = \iiint_D \nabla p\,dV,$$

$$\oiint_S \mathbf{v}(\delta\mathbf{v} \bullet \hat{\mathbf{N}})\,dS = \iiint_D \Big(\delta(\mathbf{v} \bullet \nabla)\mathbf{v} + \mathbf{v}\,\mathbf{div}\,(\delta\mathbf{v})\Big)\,dV.$$

Accordingly, we have

$$\iiint_D \left(\delta\frac{\partial\mathbf{v}}{\partial t} + \mathbf{v}\frac{\partial\delta}{\partial t} + \mathbf{v}\,\mathbf{div}\,(\delta\mathbf{v}) + \delta(\mathbf{v} \bullet \nabla)\mathbf{v} + \nabla p - \delta\mathbf{F}\right)dV = \mathbf{0}.$$

The second and third terms in the integrand cancel out by virtue of the continuity equation. Since D is arbitrary, we must therefore have

$$\delta\frac{\partial\mathbf{v}}{\partial t} + \delta(\mathbf{v} \bullet \nabla)\mathbf{v} = -\nabla p + \delta\mathbf{F}.$$

This is the **equation of motion** of the fluid. Observe that it is not a *linear* partial differential equation; the second term on the left is not linear in $\mathbf{v}$.

Electromagnetism

In 3-space there are defined two vector fields that determine the electric and magnetic forces that would be experienced by a unit charge at a particular point if it is moving with unit speed. (These vector fields are determined by electric charges and currents present in the space.) A charge q_0 at position $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ moving with velocity $\mathbf{v}_0$ experiences an electric force $q_0\mathbf{E}(\mathbf{r})$, where $\mathbf{E}$ is the **electric field**, and a magnetic force $\mu_0 q_0 \mathbf{v}_0 \times \mathbf{H}(\mathbf{r})$, where $\mathbf{H}$ is the **magnetic field** and $\mu_0 \approx 1.26 \times 10^{-6}$ N/ampere^2 is a physical constant called the **permeability of free space**. We will look briefly at each of these fields but will initially restrict ourselves to considering *static* situations. Electric fields produced by static charge distributions and magnetic fields produced by static electric currents do not depend on time. Later we will consider the interaction between the two fields when they are time dependent.

Electrostatics

Experimental evidence shows that the value of the electric field at any point $\mathbf{r}$ is the vector sum of the fields caused by any elements of charge located in 3-space. A "point charge" q at position $\mathbf{s} = \xi\mathbf{i} + \eta\mathbf{j} + \zeta\mathbf{k}$ generates the electric field

$$\mathbf{E}(\mathbf{r}) = \frac{q}{4\pi\epsilon_0}\frac{\mathbf{r}-\mathbf{s}}{|\mathbf{r}-\mathbf{s}|^3}, \qquad \text{(Coulomb's Law)},$$

where $\epsilon_0 \approx 8.85 \times 10^{-12}$ coulombs^2/N·m^2 is a physical constant called the **permittivity of free space**. This is just the field due to a point source of strength $q/4\pi\epsilon_0$ at $\mathbf{s}$. Except at $\mathbf{r} = \mathbf{s}$ the field is conservative, with potential

$$\phi(\mathbf{r}) = -\frac{q}{4\pi\epsilon_0}\frac{1}{|\mathbf{r}-\mathbf{s}|},$$

so for $\mathbf{r} \neq \mathbf{s}$ we have $\mathbf{curl}\,\mathbf{E} = \mathbf{0}$. Also $\mathbf{div}\,\mathbf{E} = 0$, except at $\mathbf{r} = \mathbf{s}$ where it is infinite; in terms of the Dirac distribution, $\mathbf{div}\,\mathbf{E} = (q/\epsilon_0)\delta(x-\xi)\delta(y-\eta)\delta(z-\zeta)$. (See Section 16.1.) The flux of $\mathbf{E}$ outward across the surface $\mathcal{S}$ of any region R containing q is

$$\oint\!\!\!\oint_{\mathcal{S}} \mathbf{E} \bullet \hat{\mathbf{N}}\, dS = \frac{q}{\epsilon_0},$$

by analogy with Example 4 of Section 16.4.

Given a *charge distribution* of density $\rho(\xi, \eta, \zeta)$ in 3-space (so that the charge in volume element $dV = d\xi\, d\eta\, d\zeta$ at $\mathbf{s}$ is $dq = \rho\, dV$), the flux of $\mathbf{E}$ out of $\mathcal{S}$ due to the charge in R is

$$\oint\!\!\!\oint_{\mathcal{S}} \mathbf{E} \bullet \hat{\mathbf{N}}\, dS = \frac{1}{\epsilon_0}\iiint_R dq = \frac{1}{\epsilon_0}\iiint_R \rho\, dV.$$

If we apply the Divergence Theorem to the surface integral, we obtain

$$\iiint_R \left(\mathbf{div}\,\mathbf{E} - \frac{\rho}{\epsilon_0}\right) dV = 0,$$

and since R is an arbitrary region,

$$\mathbf{div}\,\mathbf{E} = \frac{\rho}{\epsilon_0}.$$

This is the differential form of Gauss's Law. See Exercise 3 below.

The potential due to a charge distribution of density $\rho(\mathbf{s})$ in the region R is

$$\begin{aligned}\phi(\mathbf{r}) &= -\frac{1}{4\pi\epsilon_0}\iiint_R \frac{\rho(\mathbf{s})}{|\mathbf{r}-\mathbf{s}|}\, dV \\ &= -\frac{1}{4\pi\epsilon_0}\iiint_R \frac{\rho(\xi,\eta,\zeta)\, d\xi\, d\eta\, d\zeta}{\sqrt{(x-\xi)^2+(y-\eta)^2+(z-\zeta)^2}}.\end{aligned}$$

If ρ is continuous and vanishes outside a bounded region, the triple integral is convergent everywhere (see Exercise 4 below), so $\mathbf{E} = \nabla\phi$ is conservative throughout 3-space. Thus, at all points,

$$\textbf{curl}\,\mathbf{E} = \mathbf{0}.$$

Since $\textbf{div}\,\mathbf{E} = \textbf{div}\,\nabla\phi = \nabla^2\phi$, the potential ϕ satisfies **Poisson's equation**

$$\nabla^2\phi = \frac{\rho}{\epsilon_0}.$$

In particular, ϕ is harmonic in regions of space where no charge is distributed.

Magnetostatics

Magnetic fields are produced by moving charges, that is, by currents. Suppose that a constant electric current, I, is flowing in a filament along the curve $\mathcal{F}$. It has been determined experimentally that the magnetic field produced at position $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ by the elements of current $dI = I\,ds$ along the filament add vectorially and that the element at position $\mathbf{s} = \xi\mathbf{i} + \eta\mathbf{j} + \zeta\mathbf{k}$ produces the field

$$d\mathbf{H}(\mathbf{r}) = \frac{I}{4\pi}\frac{d\mathbf{s}\times(\mathbf{r}-\mathbf{s})}{|\mathbf{r}-\mathbf{s}|^3}, \qquad \text{(the Biot–Savart Law)},$$

where $d\mathbf{s} = \hat{\mathbf{T}}\,ds$, $\hat{\mathbf{T}}$ being the unit tangent to $\mathcal{F}$ in the direction of the current. Under the reasonable assumption that charge is not created or destroyed anywhere, the filament $\mathcal{F}$ must form a closed circuit, and the total magnetic field at $\mathbf{r}$ due to the current flowing in the circuit is

$$\mathbf{H} = \frac{I}{4\pi}\oint_{\mathcal{F}}\frac{d\mathbf{s}\times(\mathbf{r}-\mathbf{s})}{|\mathbf{r}-\mathbf{s}|^3}.$$

Let $\mathbf{A}$ be the vector field defined by

$$\mathbf{A}(\mathbf{r}) = \frac{I}{4\pi}\oint_{\mathcal{F}}\frac{d\mathbf{s}}{|\mathbf{r}-\mathbf{s}|},$$

for all $\mathbf{r}$ not on the filament $\mathcal{F}$. If we make use of the fact that

$$\nabla\left(\frac{1}{|\mathbf{r}-\mathbf{s}|}\right) = -\frac{\mathbf{r}-\mathbf{s}}{|\mathbf{r}-\mathbf{s}|^3},$$

and the vector identity $\nabla\times(\phi\mathbf{F}) = (\nabla\phi)\times\mathbf{F} + \phi(\nabla\times\mathbf{F})$ (with $\mathbf{F}$ the vector $d\mathbf{s}$, which does not depend on $\mathbf{r}$), we can calculate the curl of $\mathbf{A}$:

$$\nabla\times\mathbf{A} = \frac{I}{4\pi}\oint_{\mathcal{F}}\nabla\left(\frac{1}{|\mathbf{r}-\mathbf{s}|}\right)\times d\mathbf{s} = \frac{I}{4\pi}\oint_{\mathcal{F}} -\frac{\mathbf{r}-\mathbf{s}}{|\mathbf{r}-\mathbf{s}|^3}\times d\mathbf{s} = \mathbf{H}(\mathbf{r}).$$

Thus $\mathbf{A}$ is a vector potential for $\mathbf{H}$, and $\textbf{div}\,\mathbf{H} = 0$ at points off the filament. We can also verify by calculation that $\textbf{curl}\,\mathbf{H} = \mathbf{0}$ off the filament. (See Exercises 9–11 below.)

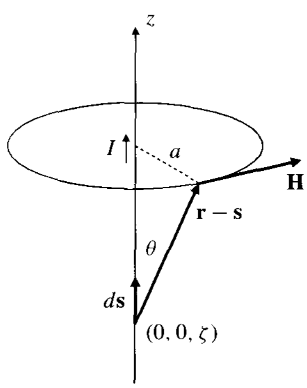

Figure 16.18

Imagine a circuit consisting of a straight filament along the z-axis with return at infinite distance. The field $\mathbf{H}$ at a finite point will then just be due to the current along the z-axis, where the current I is flowing in the direction of $\mathbf{k}$, say. The currents in all elements $d\mathbf{s}$ produce, at $\mathbf{r}$, fields in the same direction, normal to the plane containing $\mathbf{r}$ and the z-axis. (See Figure 16.18.) Therefore, the field strength $H = |\mathbf{H}|$ at a distance a from the z-axis is obtained by integrating the elements

$$dH = \frac{I}{4\pi}\frac{\sin\theta\,d\zeta}{a^2+(\zeta-z)^2} = \frac{I}{4\pi}\frac{a\,d\zeta}{\left(a^2+(\zeta-z)^2\right)^{3/2}}.$$

We have

$$\begin{aligned} H &= \frac{Ia}{4\pi}\int_{-\infty}^{\infty}\frac{d\zeta}{\left(a^2+(\zeta-z)^2\right)^{3/2}} \qquad \text{(Let } \zeta - z = a\tan\phi.\text{)} \\ &= \frac{I}{4\pi a}\int_{-\pi/2}^{\pi/2}\cos\phi\,d\phi = \frac{I}{2\pi a}. \end{aligned}$$

The field lines of $\mathbf{H}$ are evidently horizontal circles centred on the z-axis. If $\mathcal{C}_a$ is such a circle, having radius a, then the circulation of $\mathbf{H}$ around $\mathcal{C}_a$ is

$$\oint_{\mathcal{C}_a}\mathbf{H}\bullet d\mathbf{r} = \frac{I}{2\pi a}\,2\pi a = I.$$

Observe that the circulation calculated above is independent of a. In fact, if $\mathcal{C}$ is any closed curve that encircles the z-axis once counterclockwise (as seen from above), then $\mathcal{C}$ and $-\mathcal{C}_a$ comprise the oriented boundary of a washer-like surface $\mathcal{S}$ with a hole in it through which the filament passes. Since $\mathbf{curl}\,\mathbf{H} = \mathbf{0}$ on $\mathcal{S}$, Stokes's Theorem guarantees that

$$\oint_{\mathcal{C}}\mathbf{H}\bullet d\mathbf{r} = \oint_{\mathcal{C}_a}\mathbf{H}\bullet d\mathbf{r} = I.$$

Furthermore, when $\mathcal{C}$ is very small (and therefore very close to the filament), most of the contribution to the circulation of $\mathbf{H}$ around it comes from that part of the filament which is very close to $\mathcal{C}$. It therefore does not matter whether the filament is straight or infinitely long. For any closed-loop filament carrying a current, the circulation of the magnetic field around the oriented boundary of a surface through which the filament passes is equal to the current flowing in the loop. This is **Ampère's Circuital Law**. The surface is oriented with normal on the side out of which the current is flowing.

Now let us replace the filament with a more general current specified by a vector density, $\mathbf{J}$. This means that at any point $\mathbf{s}$ the current is flowing in the direction $\mathbf{J}(\mathbf{s})$ and that the current crossing an area element dS with unit normal $\hat{\mathbf{N}}$ is $\mathbf{J}\bullet\hat{\mathbf{N}}\,dS$. The circulation of $\mathbf{H}$ around the boundary $\mathcal{C}$ of surface $\mathcal{S}$ is equal to the total current flowing across $\mathcal{S}$, so

$$\oint_{\mathcal{C}}\mathbf{H}\bullet d\mathbf{r} = \iint_{\mathcal{S}}\mathbf{J}\bullet\hat{\mathbf{N}}\,dS.$$

By using Stokes's Theorem, we can replace the line integral with another surface integral and so obtain

$$\iint_{\mathcal{S}}(\mathbf{curl}\,\mathbf{H}-\mathbf{J})\bullet\hat{\mathbf{N}}\,dS = 0.$$

Since $\mathcal{S}$ is arbitrary, we must have, at all points,

$$\mathbf{curl}\,\mathbf{H} = \mathbf{J},$$

which is the pointwise version of Ampère's Circuital Law. It can be readily checked that, if

$$\mathbf{A}(\mathbf{r}) = \frac{1}{4\pi}\iiint_{R}\frac{\mathbf{J}(\mathbf{s})}{|\mathbf{r}-\mathbf{s}|}\,dV,$$

then $\mathbf{H} = \mathbf{curl}\,\mathbf{A}$ (so that $\mathbf{A}$ is a vector potential for the magnetic field $\mathbf{H}$). Here, R is the region of 3-space where $\mathbf{J}$ is nonzero. If $\mathbf{J}$ is continuous and vanishes outside a bounded set, then the triple integral converges for all $\mathbf{r}$ (Exercise 4 below), and $\mathbf{H}$ is everywhere solenoidal:

$$\mathbf{div}\,\mathbf{H} = 0.$$

Maxwell's Equations

The four equations obtained above for static electric and magnetic fields,

$$\begin{aligned} \mathbf{div}\,\mathbf{E} &= \rho/\epsilon_0 \qquad & \mathbf{div}\,\mathbf{H} &= 0 \\ \mathbf{curl}\,\mathbf{E} &= \mathbf{0} \qquad & \mathbf{curl}\,\mathbf{H} &= \mathbf{J}, \end{aligned}$$

require some modification if the fields $\mathbf{E}$ and $\mathbf{H}$ depend on time. Gauss's Law $\mathbf{div}\,\mathbf{E} = \rho/\epsilon_0$ remains valid, as does $\mathbf{div}\,\mathbf{H} = 0$, which expresses the fact that there are no *known* magnetic *sources* or *sinks* (i.e., magnetic *monopoles*). The field lines of $\mathbf{H}$ must be closed curves.

It was observed by Michael Faraday that the circulation of an electric field around a simple closed curve $\mathcal{C}$ corresponds to a change in the magnetic flux

$$\Phi = \iint_{\mathcal{S}} \mathbf{H} \bullet \hat{\mathbf{N}}\, dS$$

through any oriented surface $\mathcal{S}$ having boundary $\mathcal{C}$, according to the formula

$$\frac{d\Phi}{dt} = -\frac{1}{\mu_0} \oint_{\mathcal{C}} \mathbf{E} \bullet d\mathbf{r}.$$

Applying Stokes's Theorem to the line integral, we obtain

$$\iint_{\mathcal{S}} \mathbf{curl}\,\mathbf{E} \bullet \hat{\mathbf{N}}\, dS = \oint_{\mathcal{C}} \mathbf{E} \bullet d\mathbf{r} = -\mu_0 \frac{d}{dt} \iint_{\mathcal{S}} \mathbf{H} \bullet \hat{\mathbf{N}}\, dS = -\mu_0 \iint_{\mathcal{S}} \frac{\partial \mathbf{H}}{\partial t} \bullet \hat{\mathbf{N}}\, dS.$$

Since $\mathcal{S}$ is arbitrary, we obtain the differential form of Faraday's Law:

$$\mathbf{curl}\,\mathbf{E} = -\mu_0 \frac{\partial \mathbf{H}}{\partial t}.$$

The electric field is irrotational only if the magnetic field is constant in time.

The differential form of Ampère's Law, $\mathbf{curl}\,\mathbf{H} = \mathbf{J}$, also requires modification. If the electric field depends on time, then so will the current density $\mathbf{J}$. Assuming conservation of charge (charges are not produced or destroyed), we can show, by an argument identical to that used to obtain the continuity equation for fluid motion earlier in this section, that the rate of change of charge density satisfies

$$\frac{\partial \rho}{\partial t} = -\mathbf{div}\,\mathbf{J}.$$

(See Exercise 5 below.) This is inconsistent with Ampère's Law because $\mathbf{div}\,\mathbf{curl}\,\mathbf{H} = 0$, while $\mathbf{div}\,\mathbf{J} \neq 0$ when ρ depends on time. Note, however, that $\rho = \epsilon_0 \mathbf{div}\,\mathbf{E}$ implies that

$$-\mathbf{div}\,\mathbf{J} = \frac{\partial \rho}{\partial t} = \epsilon_0 \mathbf{div}\, \frac{\partial \mathbf{E}}{\partial t},$$

so $\mathbf{div}\left(\mathbf{J}+\epsilon_0\partial\mathbf{E}/\partial t\right)=0$. This suggests that, for the nonstatic case, Ampère's Law becomes

$$\mathbf{curl\,H}=\mathbf{J}+\epsilon_0\frac{\partial\mathbf{E}}{\partial t}$$

which indicates (as was discovered by Maxwell) that magnetic fields are not just produced by currents, but also by changing electric fields.

Together the four equations

$$\mathbf{div\,E}=\rho/\epsilon_0 \qquad \mathbf{div\,H}=0$$
$$\mathbf{curl\,E}=-\mu_0\frac{\partial\mathbf{H}}{\partial t} \qquad \mathbf{curl\,H}=\mathbf{J}+\epsilon_0\frac{\partial\mathbf{E}}{\partial t}$$

are known as **Maxwell's equations**. They govern the way electric and magnetic fields are produced in 3-space by the presence of charges and currents.

Exercises 16.6

1. **(Archimedes' principle)** A solid occupying region R with surface $\mathcal{S}$ is immersed in a liquid of constant density δ. The pressure at depth h in the liquid is δgh, so the pressure satisfies $\nabla p=\delta\mathbf{g}$, where $\mathbf{g}$ is the (vector) constant acceleration of gravity. Over each surface element dS on $\mathcal{S}$ the pressure of the fluid exerts a force $-p\hat{\mathbf{N}}\,dS$ on the solid.
   (a) Show that the resultant "buoyancy force" on the solid is
   $$\mathbf{B}=-\iiint_R \delta\mathbf{g}\,dV.$$
   Thus, the buoyancy force has the same magnitude as, and opposite direction to, the weight of the liquid displaced by the solid. This is Archimedes' principle.
   (b) Extend the above result to the case where the solid is only partly submerged in the fluid.
2. By breaking the vector $\mathbf{F}(\mathbf{G}\bullet\hat{\mathbf{N}})$ into its separate components and applying the Divergence Theorem to each separately, show that
   $$\oiint_{\mathcal{S}}\mathbf{F}(\mathbf{G}\bullet\hat{\mathbf{N}})\,dS=\iiint_D\left(\mathbf{F}\,\mathbf{div\,G}+(\mathbf{G}\bullet\nabla)\mathbf{F}\right)dV,$$
   where $\hat{\mathbf{N}}$ is the unit outward normal on the surface $\mathcal{S}$ of the domain D.
3. **(Gauss's Law)** Show that the flux of the electric field $\mathbf{E}$ outward through a closed surface $\mathcal{S}$ in 3-space is $1/\epsilon_0$ times the total charge enclosed by $\mathcal{S}$.
4. If $\mathbf{s}=\xi\mathbf{i}+\eta\mathbf{j}+\zeta\mathbf{k}$ and $f(\xi,\eta,\zeta)$ is continuous on $\mathbb{R}^3$ and vanishes outside a bounded region, show that, for any fixed $\mathbf{r}$,
   $$\iiint_{\mathbb{R}^3}\frac{|f(\xi,\eta,\zeta)|}{|\mathbf{r}-\mathbf{s}|}\,d\xi\,d\eta\,d\zeta\le\text{constant}.$$
   This shows that the potentials for the electric and magnetic fields corresponding to continuous charge and current densities that vanish outside bounded regions exist everywhere in $\mathbb{R}^3$. *Hint:* without loss of generality you can assume $\mathbf{r}=\mathbf{0}$ and use spherical coordinates.
5. The electric charge density, ρ, in 3-space depends on time as well as position if charge is moving around. The motion is described by the current density, $\mathbf{J}$. Derive the **continuity equation**
   $$\frac{\partial\rho}{\partial t}=-\mathbf{div\,J},$$
   from the fact that charge is conserved.
6. If $\mathbf{b}$ is a constant vector, show that
   $$\nabla\left(\frac{1}{|\mathbf{r}-\mathbf{b}|}\right)=-\frac{\mathbf{r}-\mathbf{b}}{|\mathbf{r}-\mathbf{b}|^3}.$$
7. If $\mathbf{a}$ and $\mathbf{b}$ are constant vectors, show that for $\mathbf{r}\neq\mathbf{b}$,
   $$\mathbf{div}\left(\mathbf{a}\times\frac{\mathbf{r}-\mathbf{b}}{|\mathbf{r}-\mathbf{b}|^3}\right)=0.$$
   Hint: use identities (d) and (h) from Theorem 3 of Section 16.2.
8. Use the result of Exercise 7 to give an alternative proof that
   $$\mathbf{div}\oint_{\mathcal{F}}\frac{d\mathbf{s}\times(\mathbf{r}-\mathbf{s})}{|\mathbf{r}-\mathbf{s}|^3}=0.$$
   Note that $\mathbf{div}$ refers to the $\mathbf{r}$ variable.

9. If $\mathbf{a}$ and $\mathbf{b}$ are constant vectors, show that for $\mathbf{r} \neq \mathbf{b}$,

$$\mathbf{curl}\left(\mathbf{a}\times\frac{\mathbf{r}-\mathbf{b}}{|\mathbf{r}-\mathbf{b}|^3}\right) = -(\mathbf{a}\bullet\nabla)\frac{\mathbf{r}-\mathbf{b}}{|\mathbf{r}-\mathbf{b}|^3}.$$

Hint: use identity (e) from Theorem 3 of Section 16.2.

10. If $\mathbf{F}$ is any smooth vector field, show that

$$\oint_{\mathcal{F}} (d\mathbf{s}\bullet\nabla)\mathbf{F}(\mathbf{s}) = \mathbf{0}$$

around any closed loop $\mathcal{F}$. *Hint:* the gradients of the components of $\mathbf{F}$ are conservative.

11. Verify that if $\mathbf{r}$ does not lie on $\mathcal{F}$, then

$$\mathbf{curl}\oint_{\mathcal{F}}\frac{d\mathbf{s}\times(\mathbf{r}-\mathbf{s})}{|\mathbf{r}-\mathbf{s}|^3} = \mathbf{0}.$$

Here, **curl** is taken with respect to the $\mathbf{r}$ variable.

12. Verify the formula $\mathbf{curl}\,\mathbf{A} = \mathbf{H}$, where $\mathbf{A}$ is the magnetic vector potential defined in terms of the steady-state current density $\mathbf{J}$.

13. If $\mathbf{A}$ is the vector potential for the magnetic field produced by a steady current in a closed-loop filament, show that $\mathbf{div}\,\mathbf{A} = 0$ off the filament.

14. If $\mathbf{A}$ is the vector potential for the magnetic field produced by a steady, continuous current density, show that $\mathbf{div}\,\mathbf{A} = 0$ everywhere. Hence, show that $\mathbf{A}$ satisfies the vector Poisson equation $\nabla^2\mathbf{A} = -\mathbf{J}$.

15. Show that in a region of space containing no charges ($\rho = 0$) and no currents ($\mathbf{J} = \mathbf{0}$), both $\mathbf{U} = \mathbf{E}$ and $\mathbf{U} = \mathbf{H}$ satisfy the wave equation

$$\frac{\partial^2\mathbf{U}}{\partial t^2} = c^2\nabla^2\mathbf{U},$$

where $c = \sqrt{1/(\epsilon_0\mu_0)} \approx 3\times 10^8$ m/s.

*** 16.** **(Heat flow in 3-space)** The heat content of a volume element dV within a homogeneous solid is $\delta cT\,dV$, where δ and c are constants (the density and specific heat of the solid material), and $T = T(x, y, z, t)$ is the temperature at time t at position (x, y, z) in the solid. Heat always flows in the direction of the negative temperature gradient and at a rate proportional to the size of that gradient. Thus, the rate of flow of heat energy across a surface element dS with normal $\hat{\mathbf{N}}$ is $-k\nabla T\bullet\hat{\mathbf{N}}\,dS$, where k is also a constant depending on the material of the solid (the coefficient of thermal conductivity). Use "conservation of heat energy" to show that for any region R with surface $\mathcal{S}$ within the solid

$$\delta c\iiint_R \frac{\partial T}{\partial t}\,dV = k\oiint_{\mathcal{S}}\nabla T\bullet\hat{\mathbf{N}}\,dS,$$

where $\hat{\mathbf{N}}$ is the unit outward normal on $\mathcal{S}$. Hence, show that heat flow within the solid is governed by the partial differential equation

$$\frac{\partial T}{\partial t} = \frac{k}{\delta c}\nabla^2 T = \frac{k}{\delta c}\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}\right).$$

16.7 Orthogonal Curvilinear Coordinates

In this optional section we will derive formulas for the gradient of a scalar field and the divergence and curl of a vector field in terms of coordinate systems more general than the Cartesian coordinate system used in the earlier sections of this chapter. In particular, we will express these quantities in terms of the cylindrical and spherical coordinate systems introduced in Section 14.6.

We denote by xyz-space the usual system of Cartesian coordinates (x, y, z) in $\mathbb{R}^3$. A different system of coordinates $[u, v, w]$ in xyz-space can be defined by a continuous transformation of the form

$$x = x(u, v, w), \qquad y = y(u, v, w), \qquad z = z(u, v, w).$$

If the transformation is one-to-one from a region D in uvw-space onto a region R in xyz-space, then a point P in R can be represented by a triple $[u, v, w]$, the (Cartiesian) coordinates of the unique point Q in uvw-space which the transformation maps to P. In this case we say that the transformation defines a **curvilinear coordinate system** in R and call $[u, v, w]$ the **curvilinear coordinates** of P with respect to that system. Note that $[u, v, w]$ are Cartesian coordinates in their own space (uvw-space); they are curvilinear coordinates in xyz-space.

Typically, we relax the requirement that the transformation defining a curvilinear coordinate system be one-to-one, that is, that every point P in R should have a unique set of curvilinear coordinates. It is reasonable to require the transformation to be only *locally one-to-one*. Thus, there may be more than one point Q that gets mapped to a point P by the transformation, but only one in any suitably small subregion of D. For example, in the plane polar coordinate system

$$x = r\cos\theta, \qquad y = r\sin\theta,$$

the transformation is locally one-to-one from D, the half of the $r\theta$-plane where $0 < r < \infty$, to the region R consisting of all points in the xy-plane except the origin. Although, say, $[1, 0]$ and $[1, 2\pi]$ are polar coordinates of the same point in the xy-plane, they are not close together in D. Observe, however, that there is still a problem with the origin, which can be represented by $[0, \theta]$ for *any* θ. Since the transformation is not even locally one-to-one at $r = 0$, we regard the origin of the xy-plane as a **singular point** for the polar coordinate system in the plane.

Example 1 The **cylindrical coordinate system** $[r, \theta, z]$ in $\mathbb{R}^3$ is defined by the transformation

$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad z = z,$$

where $r \geq 0$. (See Section 14.6.) This transformation maps the half-space D given by $r > 0$ onto all of xyz-space excluding the z-axis, and it is locally one-to-one. We regard $[r, \theta, z]$ as cylindrical polar coordinates in all of xyz-space but regard points on the z-axis as singular points of the system since the points $[0, \theta, z]$ are identical for any θ.

■

Example 2 The **spherical coordinate system** $[\rho, \phi, \theta]$ is defined by the transformation

$$x = \rho\sin\phi\cos\theta, \qquad y = \rho\sin\phi\sin\theta, \qquad z = \rho\cos\phi,$$

where $\rho \geq 0$ and $0 \leq \phi \leq \pi$. (See Section 14.6.) The transformation maps the region D in $\rho\phi\theta$-space given by $\rho > 0$, $0 < \phi < \pi$ in a locally one-to-one way onto xyz-space excluding the z-axis. The point with Cartesian coordinates $(0, 0, z)$ can be represented by the spherical coordinates $[0, \phi, \theta]$ for arbitrary ϕ and θ if $z = 0$, by $[z, 0, \theta]$ for arbitrary θ if $z > 0$, and by $[|z|, \pi, \theta]$ for arbitrary θ if $z < 0$. Thus, all points of the z-axis are singular for the spherical coordinate system.

■

Coordinate Surfaces and Coordinate Curves

Let $[u, v, w]$ be a curvilinear coordinate system in xyz-space, and let P_0 be a nonsingular point for the system. Thus, the transformation

$$x = x(u, v, w), \qquad y = y(u, v, w), \qquad z = z(u, v, w)$$

is locally one-to-one near P_0. Let P_0 have curvilinear coordinates $[u_0, v_0, w_0]$. The plane with equation $u = u_0$ in uvw-space gets mapped by the transformation to a surface in xyz-space passing through P_0. We call this surface a u-surface and still refer to it by the equation $u = u_0$; it has parametric equations

$$x = x(u_0, v, w), \qquad y = y(u_0, v, w), \qquad z = z(u_0, v, w)$$

with parameters v and w. Similarly, the v-surface $v = v_0$ and the w-surface $w = w_0$ pass through P_0; they are the images of the planes $v = v_0$ and $w = w_0$ in uvw-space.

Orthogonal curvilinear coordinates

We say that $[u, v, w]$ is an **orthogonal curvilinear coordinate system** in xyz-space if, for every nonsingular point P_0 in xyz-space, the three **coordinate surfaces** $u = u_0$, $v = v_0$, and $w = w_0$ intersect at P_0 at mutually right angles.

It is tacitly assumed that the coordinate surfaces are smooth at all nonsingular points, so we are really assuming that their normal vectors are mutually perpendicular. Figure 16.19 shows the coordinate surfaces through P_0 for a typical orthogonal curvilinear coordinate system.

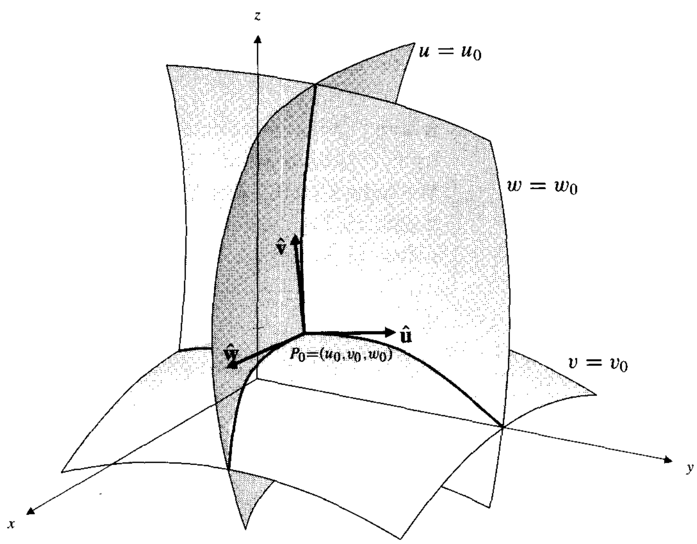

Figure 16.19 u-, v-, and w-coordinate surfaces

Pairs of coordinate surfaces through a point intersect along a **coordinate curve** through that point. For example, the coordinate surfaces $v = v_0$ and $w = w_0$ intersect along the u-**curve** with parametric equations

$$x = x(u, v_0, w_0), \quad y = y(u, v_0, w_0), \quad \text{and} \quad z = z(u, v_0, w_0),$$

where the parameter is u. A unit vector $\hat{\mathbf{u}}$ tangent to the u-curve through P_0 is normal to the coordinate surface $u = u_0$ there. Similar statements hold for unit vectors $\hat{\mathbf{v}}$ and $\hat{\mathbf{w}}$. For an orthogonal curvilinear coordinate system, the three vectors $\hat{\mathbf{u}}$, $\hat{\mathbf{v}}$, and $\hat{\mathbf{w}}$ form a basis of mutually perpendicular unit vectors at any nonsingular point P_0. (See Figure 16.19.) We call this basis the **local basis** at P_0.

Example 3 For the cylindrical coordinate system (see Figure 16.20), the coordinate surfaces are:

circular cylinders with axis along the z-axis	(r-surfaces),
vertical half-planes radiating from the z-axis	(θ-surfaces),
horizontal planes	(z-surfaces).

The coordinate curves are:

horizontal straight half-lines radiating from the z-axis	(r-curves),
horizontal circles with centres on the z-axis	(θ-curves),
vertical straight lines	(z-curves).

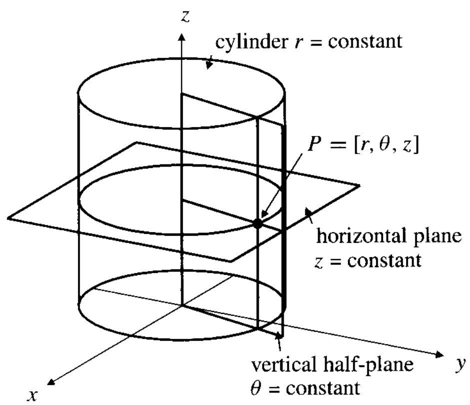

Figure 16.20 The coordinate surfaces for cylindrical coordinates

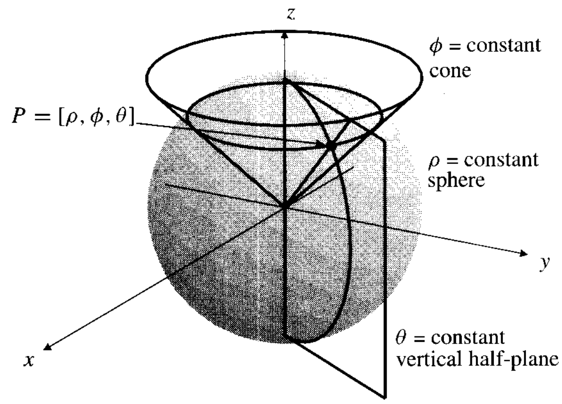

Figure 16.21 The coordinate surfaces for spherical coordinates

Example 4 For the spherical coordinate system (see Figure 16.21), the coordinate surfaces are:

spheres centred at the origin	(ρ-surfaces),
vertical circular cones with vertices at the origin	(ϕ-surfaces),
vertical half-planes radiating from the z-axis	(θ-surfaces).

The coordinate curves are:

half-lines radiating from the origin	(ρ-curves),
vertical semicircles with centres at the origin	(ϕ-curves),
horizontal circles with centres on the z-axis	(θ-curves).

Scale Factors and Differential Elements

For the rest of this section we assume that $[u, v, w]$ are **orthogonal** curvilinear coordinates in xyz-space defined via the transformation

$$x = x(u, v, w), \qquad y = y(u, v, w), \qquad z = z(u, v, w).$$

We also assume that the coordinate surfaces are smooth at any nonsingular point and that the local basis vectors $\hat{\mathbf{u}}$, $\hat{\mathbf{v}}$, and $\hat{\mathbf{w}}$ at any such point form a right-handed triad. This is the case for both cylindrical and spherical coordinates. For spherical coordinates, this is the reason we chose the order of the coordinates as $[\rho, \phi, \theta]$, rather than $[\rho, \theta, \phi]$.

The **position vector** of a point P in xyz-space can be expressed in terms of the curvilinear coordinates:

$$\mathbf{r} = x(u, v, w)\mathbf{i} + y(u, v, w)\mathbf{j} + z(u, v, w)\mathbf{k}.$$

If we hold $v = v_0$ and $w = w_0$ fixed and let u vary, then $\mathbf{r} = \mathbf{r}(u, v_0, w_0)$ defines a u-curve in xyz-space. At any point P on this curve, the vector

$$\frac{\partial \mathbf{r}}{\partial u} = \frac{\partial x}{\partial u}\mathbf{i} + \frac{\partial y}{\partial u}\mathbf{j} + \frac{\partial z}{\partial u}\mathbf{k}$$

is tangent to the u-curve at P. In general, the three vectors

$$\frac{\partial \mathbf{r}}{\partial u}, \quad \frac{\partial \mathbf{r}}{\partial v}, \quad \text{and} \quad \frac{\partial \mathbf{r}}{\partial w}$$

are tangent, respectively, to the u-curve, the v-curve, and the w-curve through P. They are also normal, respectively, to the u-surface, the v-surface, and the w-surface through P, so they are mutually perpendicular. (See Figure 16.19.) The lengths of these tangent vectors are called the *scale factors* of the coordinate system.

The **scale factors** of the orthogonal curvilinear coordinate system $[u, v, w]$ are the three functions

$$h_u = \left|\frac{\partial \mathbf{r}}{\partial u}\right|, \qquad h_v = \left|\frac{\partial \mathbf{r}}{\partial v}\right|, \qquad h_w = \left|\frac{\partial \mathbf{r}}{\partial w}\right|.$$

The scale factors are nonzero at a nonsingular point P of the coordinate system, so the local basis at P can be obtained by dividing the tangent vectors to the coordinate curves by their lengths. As noted previously, we denote the local basis vectors by $\hat{\mathbf{u}}$, $\hat{\mathbf{v}}$, and $\hat{\mathbf{w}}$. Thus,

$$\frac{\partial \mathbf{r}}{\partial u} = h_u\hat{\mathbf{u}}, \qquad \frac{\partial \mathbf{r}}{\partial v} = h_v\hat{\mathbf{v}}, \qquad \text{and} \qquad \frac{\partial \mathbf{r}}{\partial w} = h_w\hat{\mathbf{w}}.$$

The basis vectors $\hat{\mathbf{u}}$, $\hat{\mathbf{v}}$, and $\hat{\mathbf{w}}$ will form a right-handed triad provided we have chosen a suitable order for the coordinates u, v, and w.

Example 5 For cylindrical coordinates we have $\mathbf{r} = r\cos\theta\mathbf{i} + r\sin\theta\mathbf{j} + z\mathbf{k}$, so

$$\frac{\partial \mathbf{r}}{\partial r} = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}, \qquad \frac{\partial \mathbf{r}}{\partial \theta} = -r\sin\theta\,\mathbf{i} + r\cos\theta\,\mathbf{j}, \quad \text{and} \quad \frac{\partial \mathbf{r}}{\partial z} = \mathbf{k}.$$

Thus, the scale factors for the cylindrical coordinate system are given by

$$h_r = \left|\frac{\partial \mathbf{r}}{\partial r}\right| = 1, \quad h_\theta = \left|\frac{\partial \mathbf{r}}{\partial \theta}\right| = r, \quad \text{and} \quad h_z = \left|\frac{\partial \mathbf{r}}{\partial z}\right| = 1,$$

and the local basis consists of the vectors

$$\hat{\mathbf{r}} = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}, \qquad \hat{\boldsymbol{\theta}} = -\sin\theta\,\mathbf{i} + \cos\theta\,\mathbf{j}, \qquad \hat{\mathbf{z}} = \mathbf{k}.$$

See Figure 16.22. The local basis is right-handed. ■

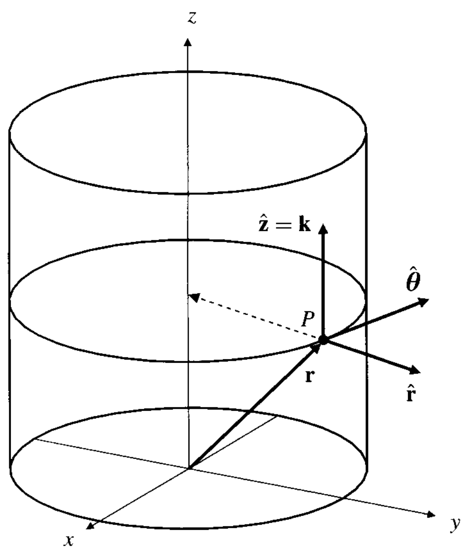

Figure 16.22 The local basis for cylindrical coordinates

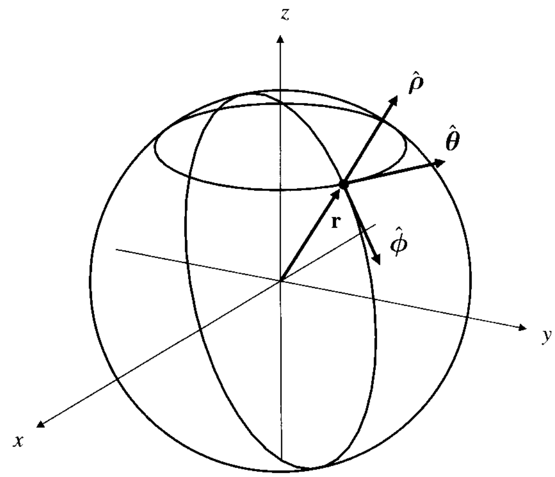

Figure 16.23 The local basis for spherical coordinates

Example 6 For spherical coordinates we have

$$\mathbf{r} = \rho \sin\phi \cos\theta \mathbf{i} + \rho \sin\phi \sin\theta \mathbf{j} + \rho \cos\phi \mathbf{k}.$$

Thus, the tangent vectors to the coordinate curves are

$$\begin{aligned}\frac{\partial \mathbf{r}}{\partial \rho} &= \sin\phi \cos\theta\, \mathbf{i} + \sin\phi \sin\theta\, \mathbf{j} + \cos\phi\, \mathbf{k},\\ \frac{\partial \mathbf{r}}{\partial \phi} &= \rho \cos\phi \cos\theta\, \mathbf{i} + \rho \cos\phi \sin\theta\, \mathbf{j} - \rho \sin\phi\, \mathbf{k},\\ \frac{\partial \mathbf{r}}{\partial \theta} &= -\rho \sin\phi \sin\theta\, \mathbf{i} + \rho \sin\phi \cos\theta\, \mathbf{j},\end{aligned}$$

and the scale factors are given by

$$h_\rho = \left|\frac{\partial \mathbf{r}}{\partial \rho}\right| = 1, \quad h_\phi = \left|\frac{\partial \mathbf{r}}{\partial \phi}\right| = \rho, \quad \text{and} \quad h_\theta = \left|\frac{\partial \mathbf{r}}{\partial \theta}\right| = \rho \sin\phi.$$

The local basis consists of the vectors

$$\begin{aligned}\hat{\boldsymbol{\rho}} &= \sin\phi \cos\theta\, \mathbf{i} + \sin\phi \sin\theta\, \mathbf{j} + \cos\phi\, \mathbf{k}\\ \hat{\boldsymbol{\phi}} &= \cos\phi \cos\theta\, \mathbf{i} + \cos\phi \sin\theta\, \mathbf{j} - \sin\phi\, \mathbf{k}\\ \hat{\boldsymbol{\theta}} &= -\sin\theta\, \mathbf{i} + \cos\theta\, \mathbf{j}.\end{aligned}$$

See Figure 16.23. The local basis is right-handed. ∎

The volume element in an orthogonal curvilinear coordinate system is the volume of an infinitesimal *coordinate box* bounded by pairs of u-, v-, and w-surfaces corresponding to values u and $u+du$, v and $v+dv$, and w and $w+dw$, respectively.

See Figure 16.24. Since these coordinate surfaces are assumed smooth, and since they intersect at right angles, the coordinate box is rectangular, and is spanned by the vectors

$$\frac{\partial \mathbf{r}}{\partial u}\,du = h_u\,du\,\hat{\mathbf{u}}, \quad \frac{\partial \mathbf{r}}{\partial v}\,dv = h_v\,dv\,\hat{\mathbf{v}}, \quad \text{and} \quad \frac{\partial \mathbf{r}}{\partial w}\,dw = h_w\,dw\,\hat{\mathbf{w}}.$$

Therefore, the volume element is given by

$$dV = h_u h_v h_w\,du\,dv\,dw.$$

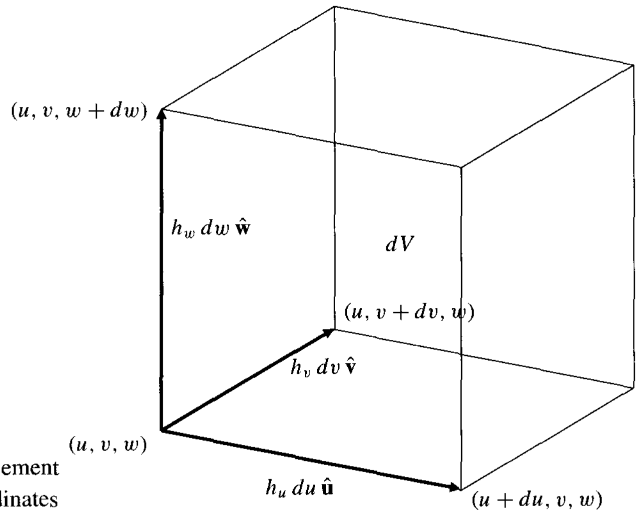

Figure 16.24 The volume element for orthogonal curvilinear coordinates

Furthermore, the surface area elements on the u-, v-, and w-surfaces are the areas of the appropriate faces of the coordinate box:

Area elements on coordinate surfaces

$$dS_u = h_v h_w\,dv\,dw, \qquad dS_v = h_u h_w\,du\,dw, \qquad dS_w = h_u h_v\,du\,dv.$$

The arc length elements along the u-, v-, and w-coordinate curves are the edges of the coordinate box:

Arc length elements on coordinate curves

$$ds_u = h_u\,du, \qquad ds_v = h_v\,dv, \qquad ds_w = h_w\,dw.$$

Example 7 For cylindrical coordinates, the volume element, as shown in Section 14.6, is

$$dV = h_r h_\theta h_z\,dr\,d\theta\,dz = r\,dr\,d\theta\,dz.$$

The surface area elements on the cylinder r = constant, the half-plane θ = constant, and the plane z = constant are, respectively,

$$dS_r = r\,d\theta\,dz, \qquad dS_\theta = dr\,dz, \qquad \text{and} \qquad dS_z = r\,dr\,d\theta.$$

■

Example 8 For spherical coordinates, the volume element, as developed in Section 14.6, is

$$dV = h_\rho h_\phi h_\theta \, d\rho \, d\phi \, d\theta = \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta.$$

The area element on the sphere ρ = constant is

$$dS_\rho = h_\phi h_\theta \, d\phi \, d\theta = \rho^2 \sin\phi \, d\phi \, d\theta.$$

The area element on the cone ϕ = constant is

$$dS_\phi = h_\rho h_\theta \, d\rho \, d\theta = \rho \sin\phi \, d\rho \, d\theta.$$

The area element on the half-plane θ = constant is

$$dS_\theta = h_\rho h_\phi \, d\rho \, d\phi = \rho \, d\rho \, d\phi.$$

■

Grad, Div, and Curl in Orthogonal Curvilinear Coordinates

The gradient ∇f of a scalar field f can be expressed in terms of the local basis at any point P with curvilinear coordinates $[u, v, w]$ in the form

$$\nabla f = F_u \hat{\mathbf{u}} + F_v \hat{\mathbf{v}} + F_w \hat{\mathbf{w}}.$$

In order to determine the coefficients F_u, F_v, and F_w in this formula, we will compare two expressions for the directional derivative of f along an arbitrary curve in xyz-space.

If the curve $\mathcal{C}$ has parametrization $\mathbf{r} = \mathbf{r}(s)$ in terms of arc length, then the directional derivative of f along $\mathcal{C}$ is given by

$$\frac{df}{ds} = \frac{\partial f}{\partial u}\frac{du}{ds} + \frac{\partial f}{\partial v}\frac{dv}{ds} + \frac{\partial f}{\partial w}\frac{dw}{ds}.$$

On the other hand, this directional derivative is also given by $df/ds = \nabla f \bullet \hat{\mathbf{T}}$, where $\hat{\mathbf{T}}$ is the unit tangent vector to $\mathcal{C}$. We have

$$\begin{aligned}\hat{\mathbf{T}} = \frac{d\mathbf{r}}{ds} &= \frac{\partial \mathbf{r}}{\partial u}\frac{du}{ds} + \frac{\partial \mathbf{r}}{\partial v}\frac{dv}{ds} + \frac{\partial \mathbf{r}}{\partial w}\frac{dw}{ds} \\ &= h_u \frac{du}{ds}\hat{\mathbf{u}} + h_v \frac{dv}{ds}\hat{\mathbf{v}} + h_w \frac{dw}{ds}\hat{\mathbf{w}}.\end{aligned}$$

Thus,

$$\frac{df}{ds} = \nabla f \bullet \hat{\mathbf{T}} = F_u h_u \frac{du}{ds} + F_v h_v \frac{dv}{ds} + F_w h_w \frac{dw}{ds}.$$

Comparing these two expressions for df/ds along $\mathcal{C}$, we see that

$$F_u h_u = \frac{\partial f}{\partial u}, \qquad F_v h_v = \frac{\partial f}{\partial v}, \qquad F_w h_w = \frac{\partial f}{\partial w}.$$

Therefore, we have shown that

The gradient in orthogonal curvilinear coordinates

$$\nabla f = \frac{1}{h_u}\frac{\partial f}{\partial u}\hat{\mathbf{u}} + \frac{1}{h_v}\frac{\partial f}{\partial v}\hat{\mathbf{v}} + \frac{1}{h_w}\frac{\partial f}{\partial w}\hat{\mathbf{w}}.$$

Example 9 In terms of cylindrical coordinates, the gradient of the scalar field $f(r, \theta, z)$ is

$$\nabla f(r, \theta, z) = \frac{\partial f}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{\partial f}{\partial z}\mathbf{k}.$$

■

Example 10 In terms of spherical coordinates, the gradient of the scalar field $f(\rho, \phi, \theta)$ is

$$\nabla f(\rho, \phi, \theta) = \frac{\partial f}{\partial \rho}\hat{\boldsymbol{\rho}} + \frac{1}{\rho}\frac{\partial f}{\partial \phi}\hat{\boldsymbol{\phi}} + \frac{1}{\rho \sin\phi}\frac{\partial f}{\partial \theta}\hat{\boldsymbol{\theta}}.$$

■

Now consider a vector field $\mathbf{F}$ expressed in terms of the curvilinear coordinates:

$$\mathbf{F}(u, v, w) = F_u(u, v, w)\hat{\mathbf{u}} + F_v(u, v, w)\hat{\mathbf{v}} + F_w(u, v, w)\hat{\mathbf{w}}.$$

The flux of $\mathbf{F}$ out of the infinitesimal coordinate box of Figure 16.24 is the sum of the fluxes of $\mathbf{F}$ out of the three pairs of opposite surfaces of the box. The flux out of the u-surfaces corresponding to u and $u + du$ is

$$\begin{aligned}
&\mathbf{F}(u + du, v, w) \bullet \hat{\mathbf{u}}\, dS_u - \mathbf{F}(u, v, w) \bullet \hat{\mathbf{u}}\, dS_u \\
&\quad = \big(F_u(u + du, v, w)h_v(u + du, v, w)h_w(u + du, v, w) \\
&\qquad - F_u(u, v, w)h_v(u, v, w)h_w(u, v, w)\big)\, dv\, dw \\
&\quad = \frac{\partial}{\partial u}\big(h_v h_w F_u\big)\, du\, dv\, dw.
\end{aligned}$$

Similar expressions hold for the fluxes out of the other pairs of coordinate surfaces.

The divergence at P of $\mathbf{F}$ is the flux *per unit volume* out of the infinitesimal coordinate box at P. Thus it is given by

The divergence in orthogonal curvilinear coordinates

$$\begin{aligned}
\mathbf{div}\,\mathbf{F}(u, v, w) = &\frac{1}{h_u h_v h_w}\bigg[\frac{\partial}{\partial u}\big(h_v h_w F_u(u, v, w)\big) \\
&+ \frac{\partial}{\partial v}\big(h_u h_w F_v(u, v, w)\big) + \frac{\partial}{\partial w}\big(h_u h_v F_w(u, v, w)\big)\bigg].
\end{aligned}$$

Example 11 For cylindrical coordinates, $h_r = h_z = 1$, and $h_\theta = r$. Thus, the divergence of $\mathbf{F} = F_r\hat{\mathbf{r}} + F_\theta\hat{\boldsymbol{\theta}} + F_z\mathbf{k}$ is

$$\begin{aligned}
\mathbf{div}\,\mathbf{F} &= \frac{1}{r}\left[\frac{\partial}{\partial r}\big(rF_r\big) + \frac{\partial}{\partial \theta}F_\theta + \frac{\partial}{\partial z}\big(rF_z\big)\right] \\
&= \frac{\partial F_r}{\partial r} + \frac{1}{r}F_r + \frac{1}{r}\frac{\partial F_\theta}{\partial \theta} + \frac{\partial F_z}{\partial z}.
\end{aligned}$$

■

Example 12 For spherical coordinates, $h_\rho = 1$, $h_\phi = \rho$, and $h_\theta = \rho\sin\phi$. The divergence of the vector field $\mathbf{F} = F_\rho\hat{\boldsymbol{\rho}} + F_\phi\hat{\boldsymbol{\phi}} + F_\theta\hat{\boldsymbol{\theta}}$ is

$$\begin{aligned}
\mathbf{div}\,\mathbf{F} &= \frac{1}{\rho^2\sin\phi}\left[\frac{\partial}{\partial\rho}(\rho^2\sin\phi\,F_\rho) + \frac{\partial}{\partial\phi}(\rho\sin\phi\,F_\phi) + \frac{\partial}{\partial\theta}(\rho\,F_\theta)\right] \\
&= \frac{1}{\rho^2}\frac{\partial}{\partial\rho}(\rho^2 F_\rho) + \frac{1}{\rho\sin\phi}\frac{\partial}{\partial\phi}(\sin\phi\,F_\phi) + \frac{1}{\rho\sin\phi}\frac{\partial F_\theta}{\partial\theta} \\
&= \frac{\partial F_\rho}{\partial\rho} + \frac{2}{\rho}F_\rho + \frac{1}{\rho}\frac{\partial F_\phi}{\partial\phi} + \frac{\cot\phi}{\rho}F_\phi + \frac{1}{\rho\sin\phi}\frac{\partial F_\theta}{\partial\theta}.
\end{aligned}$$

■

To calculate the curl of a vector fiels expressed in terms of orthogonal curvilinear coordinates we can make use of some previously obtained vector identities. First, observe that the gradient of the scalar field $f(u, v, w) = u$ is $\hat{\mathbf{u}}/h_u$, so that $\hat{\mathbf{u}} = h_u\nabla u$. Similarly, $\hat{\mathbf{v}} = h_v\nabla v$ and $\hat{\mathbf{w}} = h_w\nabla w$. Therefore, the vector field

$$\mathbf{F} = F_u\hat{\mathbf{u}} + F_v\hat{\mathbf{v}} + F_w\hat{\mathbf{w}}$$

can be written in the form

$$\mathbf{F} = F_u h_u\nabla u + F_v h_v\nabla v + F_w h_w\nabla w.$$

Using the identity $\mathbf{curl}\,(f\nabla g) = \nabla f \times \nabla g$ (see Exercise 13 of Section 16.2) we can calculate the curl of each term in the expression above. We have

$$\begin{aligned}
\mathbf{curl}\,(F_u h_u\nabla u) &= \nabla(F_u h_u)\times\nabla u \\
&= \left[\frac{1}{h_u}\frac{\partial}{\partial u}(F_u h_u)\hat{\mathbf{u}} + \frac{1}{h_v}\frac{\partial}{\partial v}(F_u h_u)\hat{\mathbf{v}} + \frac{1}{h_w}\frac{\partial}{\partial w}(F_u h_u)\hat{\mathbf{w}}\right]\times\frac{\hat{\mathbf{u}}}{h_u} \\
&= \frac{1}{h_u h_w}\frac{\partial}{\partial w}(F_u h_u)\hat{\mathbf{v}} - \frac{1}{h_u h_v}\frac{\partial}{\partial v}(F_u h_u)\hat{\mathbf{w}} \\
&= \frac{1}{h_u h_v h_w}\left[\frac{\partial}{\partial w}(F_u h_u)(h_v\hat{\mathbf{v}}) - \frac{\partial}{\partial v}(F_u h_u)(h_w\hat{\mathbf{w}})\right].
\end{aligned}$$

We have used the facts that $\hat{\mathbf{u}}\times\hat{\mathbf{u}} = \mathbf{0}$, $\hat{\mathbf{v}}\times\hat{\mathbf{u}} = -\hat{\mathbf{w}}$, and $\hat{\mathbf{w}}\times\hat{\mathbf{u}} = \hat{\mathbf{v}}$ to obtain the result above. This is why we assumed that the curvilinear coordinate system was right-handed.

Corresponding expressions can be calculated for the other two terms in the formula for $\mathbf{curl}\,\mathbf{F}$. Combining the three terms, we conclude that the curl of

$$\mathbf{F} = F_u\hat{\mathbf{u}} + F_v\hat{\mathbf{v}} + F_w\hat{\mathbf{w}}$$

is given by

The curl in orthogonal curvilinear coordinates

$$\mathbf{curl}\,\mathbf{F}(u, v, w) = \frac{1}{h_u h_v h_w}\begin{vmatrix} h_u\hat{\mathbf{u}} & h_v\hat{\mathbf{v}} & h_w\hat{\mathbf{w}} \\ \dfrac{\partial}{\partial u} & \dfrac{\partial}{\partial v} & \dfrac{\partial}{\partial w} \\ F_u h_u & F_v h_v & F_w h_w \end{vmatrix}.$$

Example 13 For cylindrical coordinates, the curl of $\mathbf{F} = F_r\hat{\mathbf{r}} + F_\theta\hat{\boldsymbol{\theta}} + F_z\mathbf{k}$ is given by

$$\begin{aligned}
\mathbf{curl}\,\mathbf{F} &= \frac{1}{r}\begin{vmatrix} \hat{\mathbf{r}} & r\hat{\boldsymbol{\theta}} & \mathbf{k} \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial \theta} & \dfrac{\partial}{\partial z} \\ F_r & rF_\theta & F_z \end{vmatrix} \\
&= \left(\frac{1}{r}\frac{\partial F_z}{\partial \theta} - \frac{\partial F_\theta}{\partial z}\right)\hat{\mathbf{r}} + \left(\frac{\partial F_r}{\partial z} - \frac{\partial F_z}{\partial r}\right)\hat{\boldsymbol{\theta}} + \left(\frac{\partial F_\theta}{\partial r} + \frac{F_\theta}{r} - \frac{1}{r}\frac{\partial F_r}{\partial \theta}\right)\mathbf{k}.
\end{aligned}$$

■

Example 14 For spherical coordinates, the curl of $\mathbf{F} = F_\rho\hat{\boldsymbol{\rho}} + F_\phi\hat{\boldsymbol{\phi}} + F_\theta\hat{\boldsymbol{\theta}}$ is given by

$$\begin{aligned}
\mathbf{curl}\,\mathbf{F} &= \frac{1}{\rho^2\sin\phi}\begin{vmatrix} \hat{\boldsymbol{\rho}} & \rho\hat{\boldsymbol{\phi}} & \rho\sin\phi\,\hat{\boldsymbol{\theta}} \\ \dfrac{\partial}{\partial \rho} & \dfrac{\partial}{\partial \phi} & \dfrac{\partial}{\partial \theta} \\ F_\rho & \rho F_\phi & \rho\sin\phi\, F_\theta \end{vmatrix} \\
&= \frac{1}{\rho\sin\phi}\left[\frac{\partial}{\partial\phi}(\sin\phi\, F_\theta) - \frac{\partial F_\phi}{\partial\theta}\right]\hat{\boldsymbol{\rho}} \\
&\quad + \frac{1}{\rho\sin\phi}\left[\frac{\partial F_\rho}{\partial\theta} - \sin\phi\frac{\partial}{\partial\rho}(\rho F_\theta)\right]\hat{\boldsymbol{\phi}} \\
&\quad + \frac{1}{\rho}\left[\frac{\partial}{\partial\rho}(\rho F_\phi) - \frac{\partial F_\rho}{\partial\phi}\right]\hat{\boldsymbol{\theta}} \\
&= \frac{1}{\rho\sin\phi}\left[(\cos\phi)F_\theta + (\sin\phi)\frac{\partial F_\theta}{\partial\phi} - \frac{\partial F_\phi}{\partial\theta}\right]\hat{\boldsymbol{\rho}} \\
&\quad + \frac{1}{\rho\sin\phi}\left[\frac{\partial F_\rho}{\partial\theta} - (\sin\phi)F_\theta - (\rho\sin\phi)\frac{\partial F_\theta}{\partial\rho}\right]\hat{\boldsymbol{\phi}} \\
&\quad + \frac{1}{\rho}\left[F_\phi + \rho\frac{\partial F_\phi}{\partial\rho} - \frac{\partial F_\rho}{\partial\phi}\right]\hat{\boldsymbol{\theta}}.
\end{aligned}$$

■

Exercises 16.7

In Exercises 1–2, calculate the gradients of the given scalar fields expressed in terms of cylindrical or spherical coordinates.

1. $f(r, \theta, z) = r\theta z$

2. $f(\rho, \phi, \theta) = \rho\phi\theta$

In Exercises 3–8, calculate **div F** and **curl F** for the given vector fields expressed in terms of cylindrical coordinates or spherical coordinates.

3. $\mathbf{F}(r, \theta, z) = r\hat{\mathbf{r}}$

4. $\mathbf{F}(r, \theta, z) = r\hat{\boldsymbol{\theta}}$

5. $\mathbf{F}(\rho, \phi, \theta) = \sin\phi\,\hat{\boldsymbol{\rho}}$

6. $\mathbf{F}(\rho, \phi, \theta) = \rho\hat{\boldsymbol{\phi}}$

7. $\mathbf{F}(\rho, \phi, \theta) = \rho\hat{\boldsymbol{\theta}}$

8. $\mathbf{F}(\rho, \phi, \theta) = \rho^2\hat{\boldsymbol{\rho}}$

9. Let $x = x(u, v)$, $y = y(u, v)$ define orthogonal curvilinear coordinates (u, v) in the xy-plane. Find the scale factors, local basis vectors, and area element for the system of coordinates (u, v).

10. Continuing the previous exercise, express the gradient of a scalar field $f(u, v)$ and the divergence and curl of a vector field $\mathbf{F}(u, v)$ in terms of the curvilinear coordinates.

11. Express the gradient of the scalar field $f(r, \theta)$ and the divergence and curl of a vector field $\mathbf{F}(r, \theta)$ in terms of plane polar coordinates (r, θ).

12. The transformation

$$x = a\cosh u\cos v, \qquad y = a\sinh u\sin v$$

defines **elliptical coordinates** in the xy-plane. This coordinate system has singular points at $x = \pm a$, $y = 0$.

(a) Show that the v-curves, u = constant, are ellipses with foci at the singular points.

(b) Show that the u-curves, v = constant, are hyperbolas with foci at the singular points.

(c) Show that the u-curve and the v-curve through a nonsingular point intersect at right angles.

(d) Find the scale factors h_u and h_v and the area element dA for the elliptical coordinate system.

13. Describe the coordinate surfaces and coordinate curves of the system of elliptical cylindrical coordinates in xyz-space defined by

$$x = a\cosh u\cos v, \qquad y = a\sinh u\sin v, \qquad z = z.$$

14. The Laplacian $\nabla^2 f$ of a scalar field f can be calculated as $\mathbf{div}\,\nabla f$. Use this method to calculate the Laplacian of the function $f(r, \theta, z)$ expressed in terms of cylindrical coordinates. (This repeats Exercise 33 of Section 14.6.)

15. Calculate the Laplacian $\nabla^2 f = \mathbf{div}\,\nabla f$ for the function $f(\rho, \phi, \theta)$, expressed in terms of spherical coordinates. (This repeats Exercise 34 of Section 14.6 but is now much easier.)

16. Calculate the Laplacian $\nabla^2 f = \mathbf{div}\,\nabla f$ for a function $f(u, v, w)$ expressed in terms of arbitrary orthogonal curvilinear coordinates (u, v, w).

Chapter Review

Key Ideas

- **What do the following terms mean?**
  - ◇ the divergence of a vector field **F**
  - ◇ the curl of a vector field **F**
  - ◇ **F** is solenoidal.
  - ◇ **F** is irrotational.
  - ◇ a scalar potential
  - ◇ a vector potential
  - ◇ orthogonal curvilinear coordinates
- **State the following theorems:**
  - ◇ the Divergence Theorem
  - ◇ Green's Theorem
  - ◇ Stokes's Theorem

Review Exercises

1. If $\mathbf{F} = x^2z\mathbf{i} + (y^2z + 3y)\mathbf{j} + x^2\mathbf{k}$, find the flux of **F** across the part of the ellipsoid $x^2 + y^2 + 4z^2 = 16$, where $z \ge 0$, oriented with upward normal.

2. Let $\mathcal{S}$ be the part of the cylinder $x^2 + y^2 = 2ax$ between the horizontal planes $z = 0$ and $z = b$, where $b > 0$. Find the flux of $\mathbf{F} = x\mathbf{i} + \cos(z^2)\mathbf{j} + e^z\mathbf{k}$ outward through $\mathcal{S}$.

3. Find $\oint_C (3y^2 + 2xe^{y^2})\,dx + (2x^2ye^{y^2})\,dy$ counterclockwise around the boundary of the parallelogram with vertices $(0, 0)$, $(2, 0)$, $(3, 1)$, and $(1, 1)$.

4. If $\mathbf{F} = -z\mathbf{i} + x\mathbf{j} + y\mathbf{k}$, what are the possible values of $\oint_C \mathbf{F} \bullet d\mathbf{r}$ around circles of radius a in the plane $2x + y + 2z = 7$?

5. Let **F** be a smooth vector field in 3-space and suppose that, for every $a > 0$, the flux of **F** out of the sphere of radius a centred at the origin is $\pi(a^3 + 2a^4)$. Find the divergence of **F** at the origin.

6. Let $\mathbf{F} = -y\mathbf{i} + x\cos(1 - x^2 - y^2)\mathbf{j} + yz\mathbf{k}$. Find the flux of $\mathbf{curl\,F}$ upward through a surface whose boundary is the curve $x^2 + y^2 = 1$, $z = 2$.

7. Let $\mathbf{F}(\mathbf{r}) = r^\lambda\mathbf{r}$, where $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and $r = |\mathbf{r}|$. For what value(s) of λ is **F** solenoidal on an open subset of 3-space? Is **F** solenoidal on all of 3-space for any value of λ?

8. Given that **F** satisfies $\mathbf{curl\,F} = \mu\mathbf{F}$ on 3-space, where μ is a nonzero constant, show that $\nabla^2\mathbf{F} + \mu^2\mathbf{F} = \mathbf{0}$.

9. Let P be a polyhedron in 3-space having n planar faces, F_1, $F_2, \ldots, F_n$. Let $\mathbf{N}_i$ be normal to F_i in the direction outward from P, and let $\mathbf{N}_i$ have length equal to the area of face F_i. Show that

$$\sum_{i=1}^{n} \mathbf{N}_i = \mathbf{0}.$$

Also, state a version of this result for a plane polygon P.

10. Around what simple, closed curve C in the xy-plane does the vector field

$$\mathbf{F} = (2y^3 - 3y + xy^2)\mathbf{i} + (x - x^3 + x^2y)\mathbf{j}$$

have the greatest circulation?

11. Through what closed, oriented surface in $\mathbb{R}^3$ does the vector field

$$\mathbf{F} = (4x + 2x^3z)\mathbf{i} - y(x^2 + z^2)\mathbf{j} - (3x^2z^2 + 4y^2z)\mathbf{k}$$

have the greatest flux?

12. Find the maximum value of

$$\oint_C \mathbf{F} \bullet d\mathbf{r},$$

where $\mathbf{F} = xy^2\mathbf{i} + (3z - xy^2)\mathbf{j} + (4y - x^2y)\mathbf{k}$, and C is a simple closed curve in the plane $x + y + z = 1$ oriented counterclockwise as seen from high on the z-axis. What curve C gives this maximum?

Challenging Problems

1. **(The expanding universe)** Let $\mathbf{v}$ be the large-scale velocity field of matter in the universe. (*Large-scale* means on the scale of intergalactic distances; *small-scale* motion such as that of planetary systems about their suns, and even stars about galactic centres, has been averaged out.) Assume that $\mathbf{v}$ is a smooth vector field. According to present astronomical theory, the distance between any two points is increasing, and the rate of increase is proportional to the distance between the points. The constant of proportionality, C, is called *Hubble's constant*. In terms of $\mathbf{v}$, if $\mathbf{r}_1$ and $\mathbf{r}_2$ are two points, then

$$\big(\mathbf{v}(\mathbf{r}_2) - \mathbf{v}(\mathbf{r}_1)\big) \bullet (\mathbf{r}_2 - \mathbf{r}_1) = C|\mathbf{r}_2 - \mathbf{r}_1|^2.$$

Show that $\mathbf{div}\,\mathbf{v}$ is constant, and find the value of the constant in terms of Hubble's constant. *Hint:* find the flux of $\mathbf{v}(\mathbf{r})$ out of a sphere of radius ϵ centred at $\mathbf{r}_1$ and take the limit as ϵ approaches zero.

2. **(Solid angle)** Two rays from a point P determine an angle at P whose measure in radians is equal to the length of the arc of the circle of radius 1 with centre at P lying between the two rays. Similarly, an arbitrarily shaped half-cone K with vertex at P determines a **solid angle** at P whose measure in **steradians** (*stereo* + *radians*) is the area of that part of the sphere of radius 1 with centre at P lying within K. For example, the first octant of $\mathbb{R}^3$ is a half-cone with vertex at the origin. It determines a solid angle at the origin measuring

$$4\pi \times \frac{1}{8} = \frac{\pi}{2} \text{ steradians},$$

since the area of the unit sphere is 4π. (See Figure 16.25.)

(a) Find the steradian measure of the solid angle at the vertex of a right-circular half-cone whose generators make angle α with its central axis.

(b) If a smooth, oriented surface intersects the general half-cone K but not at its vertex P, let S be the part of the surface lying within K. Orient S with normal pointing away from P. Show that the steradian measure of the solid angle at P determined by K is the flux of $\mathbf{r}/|\mathbf{r}|^3$ through S, where $\mathbf{r}$ is the vector from P to the point (x, y, z).

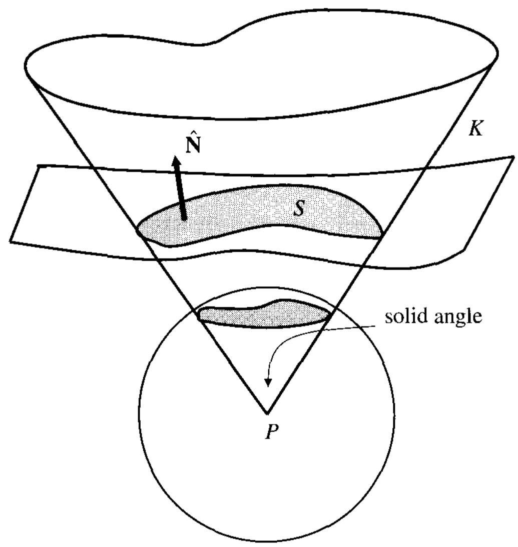

Figure 16.25

Integrals over moving domains

By the Fundamental Theorem of Calculus, the derivative with respect to time t of an integral of $f(x, t)$ over a "moving interval" $[a(t), b(t)]$ is given by

$$\frac{d}{dt}\int_{a(t)}^{b(t)} f(x,t)\,dx = \int_{a(t)}^{b(t)} \frac{\partial}{\partial t} f(x,t)\,dx + f(b(t),t)\frac{db}{dt} - f(a(t),t)\frac{da}{dt}.$$

The next three problems, suggested by Luigi Quartapelle of the Politecnico di Milano, provide various extensions of this one-dimensional result to higher dimensions. The calculations are somewhat lengthy, so you may want to try to get some help from a computer algebra system.

* 3. **(Rate of change of circulation along a moving curve)**

(a) Let $\mathbf{F}(\mathbf{r}, t)$ be a smooth vector field in $\mathbb{R}^3$ depending on a parameter t, and let

$$\mathbf{G}(s,t) = \mathbf{F}\big(\mathbf{r}(s,t), t\big) = \mathbf{F}\big(x(s,t), y(s,t), z(x,t), t\big),$$

where $\mathbf{r}(s,t) = x(s,t)\mathbf{i} + y(s,t)\mathbf{j} + z(s,t)\mathbf{k}$ has continuous partial derivatives of second order. Show that

$$\frac{\partial}{\partial t}\left(\mathbf{G} \bullet \frac{\partial \mathbf{r}}{\partial s}\right) - \frac{\partial}{\partial s}\left(\mathbf{G} \bullet \frac{\partial \mathbf{r}}{\partial t}\right) = \frac{\partial \mathbf{F}}{\partial t} \bullet \frac{\partial \mathbf{r}}{\partial s} + \left((\nabla \times \mathbf{F}) \times \frac{\partial \mathbf{r}}{\partial t}\right) \bullet \frac{\partial \mathbf{r}}{\partial s}.$$

Here, the curl $\nabla \times \mathbf{F}$ is taken with respect to the position vector $\mathbf{r}$.

(b) For fixed t (which you can think of as time), $\mathbf{r} = \mathbf{r}(s, t)$, $(a \le s \le b)$, represents parametrically a curve C_t in $\mathbb{R}^3$. The curve moves as t varies; the velocity of any point on C_t is $\mathbf{v}_C(s, t) = \partial\mathbf{r}/\partial t$. Show that

$$\frac{d}{dt}\int_{C_t} \mathbf{F} \bullet d\mathbf{r} = \int_{C_t} \frac{\partial \mathbf{F}}{\partial t} \bullet d\mathbf{r} + \int_{C_t} \big((\nabla \times \mathbf{F}) \times \mathbf{v}_C\big) \bullet d\mathbf{r} + \mathbf{F}\big(\mathbf{r}(b, t), t\big) \bullet \mathbf{v}_C(b, t) - \mathbf{F}\big(\mathbf{r}(a, t), t\big) \bullet \mathbf{v}_C(a, t).$$

Hint: write

$$\begin{aligned}\frac{d}{dt}\int_{C_t} \mathbf{F} \bullet d\mathbf{r} &= \int_a^b \frac{\partial}{\partial t}\left(\mathbf{G} \bullet \frac{\partial \mathbf{r}}{\partial s}\right) ds \\ &= \int_a^b \left[\frac{\partial}{\partial s}\left(\mathbf{G} \bullet \frac{\partial \mathbf{r}}{\partial t}\right) + \left(\frac{\partial}{\partial t}\left(\mathbf{G} \bullet \frac{\partial \mathbf{r}}{\partial s}\right) - \frac{\partial}{\partial s}\left(\mathbf{G} \bullet \frac{\partial \mathbf{r}}{\partial t}\right)\right)\right] ds.\end{aligned}$$

Now use the result of (a).

* **4. (Rate of change of flux through a moving surface)** Let S_t be a moving surface in $\mathbb{R}^3$ smoothly parametrized (for each t) by

$$\mathbf{r} = \mathbf{r}(u, v, t) = x(u, v, t)\mathbf{i} + y(u, v, t)\mathbf{j} + z(u, v, t)\mathbf{k},$$

where (u, v) belongs to a parameter region R in the uv-plane. Let $\mathbf{F}(\mathbf{r}, t) = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$ be a smooth 3-vector function, and let $\mathbf{G}(u, v, t) = \mathbf{F}(\mathbf{r}(u, v, t), t)$.

(a) Show that

$$\begin{aligned}&\frac{\partial}{\partial t}\left(\mathbf{G} \bullet \left[\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\right]\right) - \frac{\partial}{\partial u}\left(\mathbf{G} \bullet \left[\frac{\partial \mathbf{r}}{\partial t} \times \frac{\partial \mathbf{r}}{\partial v}\right]\right) - \frac{\partial}{\partial v}\left(\mathbf{G} \bullet \left[\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial t}\right]\right) \\ &= \frac{\partial \mathbf{F}}{\partial t} \bullet \left[\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\right] + (\nabla \bullet \mathbf{F})\frac{\partial \mathbf{r}}{\partial t} \bullet \left[\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\right].\end{aligned}$$

(b) If C_t is the boundary of S_t with orientation corresponding to that of S_t, use Green's Theorem to show that

$$\iint_R \left[\frac{\partial}{\partial u}\left(\mathbf{G} \bullet \left[\frac{\partial \mathbf{r}}{\partial t} \times \frac{\partial \mathbf{r}}{\partial v}\right]\right) + \frac{\partial}{\partial v}\left(\mathbf{G} \bullet \left[\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial t}\right]\right)\right] du\,dv = \oint_{C_t} \left(\mathbf{F} \times \frac{\partial \mathbf{r}}{\partial t}\right) \bullet d\mathbf{r}.$$

(c) Combine the results of (a) and (b) to show that

$$\begin{aligned}&\frac{d}{dt}\iint_{S_t} \mathbf{F} \bullet \hat{\mathbf{N}}\, dS \\ &= \iint_{S_t} \frac{\partial \mathbf{F}}{\partial t} \bullet \hat{\mathbf{N}}\, dS + \iint_{S_t} (\nabla \bullet \mathbf{F})\mathbf{v}_S \bullet \hat{\mathbf{N}}\, dS + \oint_{C_t} (\mathbf{F} \times \mathbf{v}_C) \bullet d\mathbf{r},\end{aligned}$$

where $\mathbf{v}_S = \partial\mathbf{r}/\partial t$ on S_t is the velocity of S_t, $\mathbf{v}_C = \partial\mathbf{r}/\partial t$ on C_t is the velocity of C_t, and $\hat{\mathbf{N}}$ is the unit normal field on S_t corresponding to its orientation.

* **5. (Rate of change of integrals over moving volumes)** Let S_t be the position at time t of a smooth, closed surface in $\mathbb{R}^3$ that varies smoothly with t and bounds at any time t a region D_t. If $\hat{\mathbf{N}}(\mathbf{r}, t)$ denotes the unit outward (from D_t) normal field on S_t, and $\mathbf{v}_S(\mathbf{r}, t)$ is the velocity of the point $\mathbf{r}$ on S_t at time t, show that

$$\frac{d}{dt}\iiint_{D_t} f\, dV = \iiint_{D_t} \frac{\partial f}{\partial t}\, dV + \oiint_{S_t} f\mathbf{v}_S \bullet \hat{\mathbf{N}}\, dS$$

holds for smooth functions $f(\mathbf{r}, t)$. *Hint:* let ΔD_t consist of the points through which S_t passes as t increases to $t + \Delta t$. The volume element dV in ΔD_t can be expressed in terms of the area element dS on S_t by

$$dV = \mathbf{v} \bullet \hat{\mathbf{N}}\, dS\, \Delta t.$$

Show that

$$\begin{aligned}&\frac{1}{\Delta t}\left[\iiint_{D_{t+\Delta t}} f(\mathbf{r}, t + \Delta t)\, dV - \iiint_{D_t} f(\mathbf{r}, t)\, dV\right] \\ &= \iiint_{D_t} \frac{f(\mathbf{r}, t + \Delta t) - f(\mathbf{r}, t)}{\Delta t}\, dV + \frac{1}{\Delta t}\iiint_{\Delta D_t} f(\mathbf{r}, t)\, dV + \iiint_{\Delta D_t} \frac{f(\mathbf{r}, t + \Delta t) - f(\mathbf{r}, t)}{\Delta t}\, dV,\end{aligned}$$

and show that the last integral $\to 0$ as $\Delta t \to 0$.